# Space Electronic Reconnaissance Localization Theories and Methods 

Fucheng Guo | Yun Fan | Yiyu Zhou | Caigen Zhou | Qiang Li



## SPACE ELECTRONIC RECONNAISSANCE

# SPACE ELECTRONIC RECONNAISSANCE LOCALIZATION THEORIES AND METHODS 

Fucheng Guo<br>National University of Defense Technology, P.R. China

Yun Fan
National University of Defense Technology, P.R. China
Yiyu Zhou
National University of Defense Technology, P.R. China
Caigen Zhou
National University of Defense Technology, P.R. China
Qiang Li
National University of Defense Technology, P.R. China

Registered office
John Wiley \& Sons Singapore Pte. Ltd., 1 Fusionopolis Walk, \#07-01 Solaris South Tower, Singapore 138628.
For details of our global editorial offices, for customer services and for information about how to apply for permission to reuse the copyright material in this book please see our website at www.wiley.com.

All Rights Reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, scanning, or otherwise, except as expressly permitted by law, without either the prior written permission of the Publisher, or authorization through payment of the appropriate photocopy fee to the Copyright Clearance Center. Requests for permission should be addressed to the Publisher, John Wiley \& Sons Singapore Pte. Ltd., 1 Fusionopolis Walk, \#07-01 Solaris South Tower, Singapore 138628, tel: 65-66438000, fax: 65-66438008, email: enquiry @ wiley.com.

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print may not be available in electronic books.

Designations used by companies to distinguish their products are often claimed as trademarks. All brand names and product names used in this book are trade names, service marks, trademarks or registered trademarks of their respective owners. The Publisher is not associated with any product or vendor mentioned in this book. This publication is designed to provide accurate and authoritative information in regard to the subject matter covered. It is sold on the understanding that the Publisher is not engaged in rendering professional services. If professional advice or other expert assistance is required, the services of a competent professional should be sought.

MATLAB ${ }^{\circledR}$ is a trademark of The MathWorks, Inc. and is used with permission. The MathWorks does not warrant the accuracy of the text or exercises in this book. This book's use or discussion of MATLAB ${ }^{\circledR}$ software or related products does not constitute endorsement or sponsorship by The MathWorks of a particular pedagogical approach or particular use of the MATLAB ${ }^{\circledR}$ software.

Limit of Liability/Disclaimer of Warranty: While the publisher and author have used their best efforts in preparing this book, they make no representations or warranties with respect to the accuracy or completeness of the contents of this book and specifically disclaim any implied warranties of merchantability or fitness for a particular purpose. It is sold on the understanding that the publisher is not engaged in rendering professional services and neither the publisher nor the author shall be liable for damages arising herefrom. If professional advice or other expert assistance is required, the services of a competent professional should be sought.

## Library of Congress Cataloging-in-Publication Data applied for.

A catalogue record for this book is available from the British Library.
ISBN: 978-1-118-54219-4
Set in 10/12pt Times by Laserwords Private Limited, Chennai, India

To those anonymous engineers who dedicated their lives to the China national defense industry

## Contents

Preface ..... xiii
Acknowledgments ..... XV
Acronyms ..... xvii
1 Introduction to Space Electronic Reconnaissance Geolocation ..... 1
1.1 Introduction ..... 1
1.2 An Overview of Space Electronic Reconnaissance Geolocation Technology ..... 3
1.2.1 Geolocation of an Emitter on the Earth ..... 3
1.2.2 Tracking of an Emitter on a Satellite ..... 8
1.2.3 Geolocation by Near-Space Platforms ..... 9
1.3 Structure of a Typical SER System ..... 9
References ..... 11
2 Fundamentals of Satellite Orbit and Geolocation ..... 13
2.1 An Introduction to the Satellite and Its Orbit ..... 13
2.1.1 Kepler's Three Laws ..... 13
2.1.2 Classification of Satellite Orbits ..... 15
2.2 Orbit Parameters and State of Satellite ..... 18
2.2.1 Orbit Elements of a Satellite ..... 18
2.2.2 Definition of Several Arguments of Perigee and Their Correlations ..... 20
2.3 Definition of Coordinate Systems and Their Transformations ..... 21
2.3.1 Definition of Coordinate Systems ..... 21
2.3.2 Transformation between Coordinate Systems ..... 25
2.4 Spherical Model of the Earth for Geolocation ..... 27
2.4.1 Regular Spherical Model for Geolocation ..... 27
2.4.2 Ellipsoid Model of the Earth ..... 27
2.5 Coverage Area of a Satellite ..... 30
2.5.1 Approximate Calculation Method for the Coverage Area ..... 30
2.5.2 Examples of Calculation of the Coverage Area ..... 31
2.5.3 Side Reconnaissance Coverage Area ..... 33
2.6 Fundamentals of Geolocation ..... 33
2.6.1 Spatial Geolocation Plane ..... 34
2.6.2 Spatial Line of Position (LOP) ..... 34
2.7 Measurement Index of Geolocation Errors ..... 38
2.7.1 General Definition of Error ..... 38
2.7.2 Geometrical Dilution of Precision (GDOP) ..... 40
2.7.3 Graphical Representation of the Geolocation Error ..... 40
2.7.4 Spherical Error Probability (SEP) and Circular Error Probability (CEP) ..... 41
2.8 Observability Analysis of Geolocation ..... 44
References ..... 45
3 Single-Satellite Geolocation System Based on Direction Finding ..... 47
3.1 Direction Finding Techniques ..... 47
3.1.1 Amplitude Comparison DF Technique ..... 48
3.1.2 Interferometer DF Technique ..... 49
3.1.3 Array-Based DF Technique ..... 55
3.1.4 Other DF Techniques ..... 57
3.2 Single-Satellite LOS Geolocation Method and Analysis ..... 57
3.2.1 Model of LOS Geolocation ..... 57
3.2.2 Solution of LOS Geolocation ..... 59
3.2.3 CRLB of the LOS Geolocation Error ..... 60
3.2.4 Simulation and Analysis of the LOS Geolocation Error ..... 62
3.2.5 Geometric Distribution of the LOS Geolocation Error ..... 63
3.3 Multitimes Statistic LOS Geolocation ..... 64
3.3.1 Single-Satellite Multitimes Triangulation ..... 65
3.3.2 Average for Single-Satellite Multitimes Geolocation ..... 66
3.3.3 Weighted Average for Single-Satellite Multitimes Geolocation ..... 67
3.3.4 Simulation of Single-Satellite LOS Geolocation ..... 67
3.4 Single HEO Satellite LOS Geolocation ..... 73
3.4.1 Analysis of Single GEO Satellite LOS Geolocation ..... 73
3.4.2 Geosynchronous Satellite Multitimes LOS Geolocation ..... 74
References ..... 77
4 Multiple Satellites Geolocation Based on TDOA Measurement ..... 79
4.1 Three-Satellite Geolocation Based on a Regular Sphere ..... 80
4.1.1 Three-Satellite Geolocation Solution Method ..... 80
4.1.2 Multisatellite TDOA Geolocation Method ..... 82
4.1.3 CRLB of a Multisatellite TDOA Geolocation Error ..... 85
4.1.4 Osculation Error of the Spherical Earth Model ..... 86
4.2 Three-Satellite Geolocation Based on the WGS-84 Earth Surface Model ..... 88
4.2.1 Analytical Method ..... 89
4.2.2 Spherical Iteration Method ..... 92
4.2.3 Newton Iteration Method ..... 94
4.2.4 Performance Comparison among the Three Solution Methods ..... 96
4.2.5 Altitude Input Location Algorithm ..... 100
4.3 Ambiguity and No-Solution Problems of Geolocation ..... 102
4.3.1 Ambiguity Problem of Geolocation ..... 102
4.3.2 No-Solution Problem of Geolocation ..... 106
4.4 Error Analysis of Three-Satellite Geolocation ..... 109
4.4.1 Analysis of the Random Geolocation Error ..... 109
4.4.2 Analysis of Bias Caused by Altitude Assumption ..... 112
4.4.3 Influence of Change of the Constellation Geometric Configuration on GDOP ..... 114
4.5 Calibration Method of the Three-Satellite TDOA Geolocation System ..... 117
4.5.1 Four-Station Calibration Method and Analysis ..... 117
4.5.2 Three-Station Calibration Method ..... 125
References ..... 130
5 Dual-Satellite Geolocation Based on TDOA and FDOA ..... 133
5.1 Introduction of TDOA-FDOA Geolocation by a Dual-Satellite ..... 133
5.1.1 Explanation of Dual-Satellite Geolocation Theory ..... 133
5.1.2 Structure of Dual-Satellite TDOA-FDOA Geolocation System ..... 134
5.2 Dual LEO Satellite TDOA-FDOA Geolocation Method ..... 136
5.2.1 Geolocation Model ..... 136
5.2.2 Solution Method of Algebraic Analysis ..... 138
5.2.3 Approximate Analytical Method for Same-Orbit Satellites ..... 141
5.2.4 Method for Eliminating an Ambiguous Geolocation Point ..... 143
5.3 Error Analysis for TDOA-FDOA Geolocation ..... 144
5.3.1 Analytic Method for the Geolocation Error ..... 144
5.3.2 GDOP of the Dual LEO Satellite Geolocation Error ..... 146
5.3.3 Analysis of Various Factors Influencing GDOP ..... 151
5.4 Dual HEO Satellite TDOA-FDOA Geolocation ..... 152
5.4.1 Dual Geosynchronous Orbit Satellites TDOA-FDOA Geolocation ..... 152
5.4.2 Calibration Method Based on Reference Sources ..... 155
5.4.3 Calibration Method Using Multiple Reference Sources ..... 159
5.4.4 Flow of Calibration and Geolocation ..... 164
5.5 Method of Measuring TDOA and FDOA ..... 165
5.5.1 The Cross-Ambiguity Function ..... 165
5.5.2 Theoretical Analysis on the TDOA-FDOA Measurement Performance ..... 166
5.5.3 Segment Correlation Accumulation Method for CAF Computation ..... 168
5.5.4 Resolution of Multiple Signals of the Same Time and Same Frequency ..... 172
References ..... 174
6 Single-Satellite Geolocation System Based on the Kinematic Principle ..... 177
6.1 Single-Satellite Geolocation Model ..... 177
6.2 Single-Satellite Single-Antenna Frequency-Only Based Geolocation ..... 179
6.2.1 Frequency-Only Based Geolocation Method ..... 179
6.2.2 Analysis of the Geolocation Error ..... 180
6.2.3 Analysis of the Frequency-Only Based Geolocation Error ..... 181
6.3 Single-Satellite Geolocation by the Frequency Changing Rate Only ..... 183
6.3.1 Model of Geolocation by the Frequency Changing Rate Only ..... 183
6.3.2 CRLB of the Geolocation Error ..... 185
6.3.3 Geolocation Simulation ..... 186
6.4 Single-Satellite Single-Antenna TOA-Only Geolocation ..... 186
6.4.1 Model and Method of TOA-Only Geolocation ..... 186
6.4.2 Analysis of the Geolocation Error ..... 189
6.4.3 Geolocation Simulation ..... 192
6.5 Single-Satellite Interferometer Phase Rate of Changing-Only Geolocation ..... 192
6.5.1 Geolocation Model ..... 192
6.5.2 Geolocation Algorithm ..... 195
6.5.3 CRLB of the Geolocation Error ..... 196
6.5.4 Calculation Analysis of the Geolocation Error ..... 197
References ..... 201
7 Geolocation by Near-Space Platforms ..... 203
7.1 An Overview of Geolocation by Near-Space Platforms ..... 203
7.1.1 Near-Space Platform Overview ..... 203
7.1.2 Geolocation by the Near-Space Platform ..... 204
7.2 Multiplatform Triangulation ..... 204
7.2.1 Theory of 2D Triangulation ..... 204
7.2.2 Error Analysis for Dual-Station Triangulation ..... 205
7.2.3 Optimal Geometric Configuration of Observers ..... 207
7.3 Multiplatform TDOA Geolocation ..... 211
7.3.1 Theory of Multiplatform TDOA Geolocation ..... 211
7.3.2 2D TDOA Geolocation Algorithm ..... 212
7.3.3 TDOA Geolocation Using the Altitude Assumption ..... 215
7.3.4 3D TDOA Geolocation Algorithm ..... 215
7.4 Localization Theory by a Single Platform ..... 217
7.4.1 Measurement Model of Localization ..... 218
7.4.2 A 2D Approximate Localization Method ..... 219
7.4.3 MGEKF (Modified Gain Extended Kalman Filter) Localization Method ..... 221
7.4.4 Simulation ..... 223
References ..... 225
8 Satellite-to-Satellite Passive Orbit Determination by Bearings Only ..... 227
8.1 Introduction ..... 227
8.2 Model and Method of Bearings-Only Passive Tracking ..... 227
8.2.1 Mathematic Model in the Case of the Two-Body Problem ..... 228
8.2.2 Tracking Method in the Case of the Two-Body Model ..... 229
8.2.3 Mathematical Model Considering $J_{2}$ Perturbation of Earth Oblateness ..... 232
8.2.4 Tracking Method Considering $\boldsymbol{J}_{2}$ Perturbation of Earth Oblateness ..... 233
8.3 System Observability Analysis ..... 235
8.3.1 Description Method for System Observability ..... 235
8.3.2 Influence of Factors on the State Equation ..... 236
8.3.3 Influence of Factors on the Measurement Equation ..... 237
8.4 Tracking Simulation and Analysis ..... 239
8.4.1 Simulation in the Case of the Two-Body Model ..... 241
8.4.2 Simulation Considering $J_{2}$ Perturbation of Earth Oblateness ..... 251
8.5 Summary ..... 258
References ..... 259
$9 \quad$ Satellite-to-Satellite Passive Tracking Based on Angle and Frequency Information ..... 261
9.1 Introduction of Passive Tracking ..... 261
9.2 Tracking Model and Method ..... 262
9.2.1 Mathematic Model in the Case of the Two-Body Model ..... 262
9.2.2 Tracking Method in the Case of the Two-Body Model ..... 263
9.2.3 Mathematical Models Considering $J_{2}$ Perturbation of Earth Oblateness ..... 266
9.2.4 Tracking Method Considering $\boldsymbol{J}_{2}$ Perturbation of Earth Oblateness ..... 267
9.3 System Observability Analysis ..... 268
9.3.1 Influence of Factors of the State Equation ..... 269
9.3.2 Influence of Factors of the Measurement Equation ..... 269
9.4 Simulation and Its Analysis ..... 277
9.4.1 Simulation in the Case of the Two-Body Model ..... 278
9.4.2 Simulation Considering $J_{2}$ Perturbation of Earth Oblateness ..... 296
9.5 Summary ..... 308
References ..... 309
10 Satellite-to-Satellite Passive Orbit Determination Based on Frequency Only ..... 311
10.1 The Theory and Mathematical Model of Passive Orbit Determination Based on Frequency Only ..... 313
10.1.1 The Theory of Orbit Determination Based on Frequency Only ..... 313
10.1.2 The System Model in the Case of the Two-Body Model ..... 313
10.1.3 The System Model for $J_{2}$ Perturbation of Earth Oblateness ..... 315
10.2 Satellite-to-Satellite Passive Orbit Determination Based on PSO and Frequency ..... 317
10.2.1 Introduction of Particle Swarm Optimization (PSO) ..... 317
10.2.2 Orbit Determination Method Based on the PSO Algorithm ..... 319
10.3 System Observability Analysis ..... 320
10.3.1 Simulation Scenario 1 ..... 322
10.3.2 Simulation Scenario 2 ..... 323
10.3.3 Simulation Scenario 3 ..... 325
10.4 CRLB of the Orbit Parameter Estimation Error ..... 329
10.5 Orbit Determination and Tracking Simulation and Its Analysis ..... 333
10.5.1 Simulation in the Case of the Two-Body Model ..... 334
10.5.2 Simulation in the Case of Considering the Perturbation ..... 347
References ..... 348
11 A Prospect of Space Electronic Reconnaissance Technology ..... 349
Appendix Transformation of Orbit Elements, State and Coordinates of Satellites in Two-Body Motion ..... 351
Index ..... 355

## Preface

With the development of aerospace technology, information technology, and electronic warfare (EW) technology in last few decades, space electronic reconnaissance (SER) technology has drawn great attention for its wide coverage and full time, 24/7 interception of a transmitting source providing electronic intelligence (ELINT), communication intelligence (COMINT), or signal intelligence (SIGINT). Various electronic reconnaissance systems have been developed by the United States, Russia, Japan, and the European Union and their importance has been noticed in recent conflicts. Not surprisingly, China has also made great progress in this field in recent years. In the process of electronic reconnaissance, one of the crucial tasks is to locate the transmitting source, or the transmitter on the earth or in space. Due to the motion of the satellite in an orbit and the relatively high altitude of the reconnaissance platform, the SER system differs greatly from the traditional passive detection and location system on land and in oceans in terms of geolocation theory, method, and system realization. Therefore, it is rather meaningful and useful to research the geolocation theory and the method for the SER system.
As the SER system is mainly used for military intelligence (such as ELINT, COMINT, and SIGINT), early warning, battlefield awareness, and electromagnetic spectrum survey, the relevant technologies are always confidential so it was rare to find detailed technological literature. However, there is still some theoretic or technological literature on SER geolocation, which are dispersed among different reports, journal papers, and books, but until now there has been no academic book on the SER geolocation technologies, which is far from being commensurate with the current ever-increasing development in this field. Therefore, after organizing the reports and papers written by our research group in this field in the last decade and some of the relevant technological literature, we wrote this book, which covers theory and methods on SER geolocation.
To introduce the theory and methods on SER geolocation systematically, this book covers the development of concepts, theories, technologies, and methods on SER geolocation over the last decade. Firstly, the concept and system of SER geolocation are introduced. The geolocation theory by a single satellite based on the line-of-sight (LOS) information, which is measured by the direction finding (DF) system, was discussed. Then the geolocation theory by multiple satellites based on the time difference of arrival (TDOA), geolocation by dual-satellites based on TDOA and frequency difference of arrival (FDOA), geolocation by a single satellite
based on particle kinematics and geolocation by near-space platform for geolocation ground transmitters were introduced in detail. At the same time, the orbit determination problem of a satellite using DF and frequency information by an aerospace platform in deep space was analyzed and explored.
There are 11 chapters in this book: an introduction of SER localization technology, knowledge about the satellite orbit and basic terminology of geolocation, single-satellite geolocation technology based on DF, three-satellite geolocation technology based on TDOA, two-satellite geolocation technology based on TDOA and FDOA, the single-satellite localization technology based on kinematics theory, localization principles of near-space platform electronic reconnaissance systems, the orbit determination of single satellite-to-satellite tracking using bearings only ( BO ) information, the orbit determination of single satellite-to-satellite tracking using bearings and frequency information, and the orbit determination of single satellite-to-satellite tracking using frequency only (FO) information. At the end of the book, the perspective of the SER technology is given.
This book might be helpful to engineers who are researching the space information countermeasurement, the aerospace application system, EW, intelligence reconnaissance, and the signal and data processing system. It might also be useful to graduate students or teachers researching aerospace science and technology, information and communication engineering, electrical engineering in university or college, or the administration officers in the defense industry and military officers in the army.

## Acknowledgments

Part of this work was funded by the National Natural Science Foundation of China (NFSC) Project No. 60702010. Most of this book is based on some papers, technological research reports by authors and their dissertations, and theses of students in our research group. Many thanks should be given to them.
In the process of writing this book, we were supported by colleagues of the School of Electronic Science and Engineering, the National University of Defense Technology (NUDT), and the Institute of Northern China Electronic Equipment. Special thanks should be given to Prof. Jiang Wenli, Prof. Deng Xinpu, Associate Prof. Liu Zheng, Dr Xu Dan, Dr Zhong Danxing, Dr Wang Qiang, Dr Sheng Weidong, Dr Jia Xinjiang, Dr Li Teng, Mr Miao Yu, Mr Gao Qian, and Ms Zheng Jin for their relevant technological materials and ideas. We also thank Prof. Xu Hui, Prof. An Wei, Prof. Wu Jing, Prof. Huang Zhitao, Associate Prof. Feng Daowang, Dr Xie Kai, Dr Liu Haijun, Dr Han Tao, Mr Peng Feng, Dr Xu Zhan, Dr Xu Yi, Dr Zhang Min, Mr Liu Xiaoguang, and Dr Li Jinzhou for their dedicated efforts and Prof. Mao Shiyi and Prof. Wei Jibo for their valuable suggestions and comments.
The authors hereby also acknowledge supervisor Prof. Sun Zhongkang for his kindly direction and support over many years in my PhD research area of passive location and tracking technologies. Many thanks should be given to T-Win Translation Company which provided wonderful translation from Chinese to English.

## Acronyms

| 2D | Two-dimensional |
| :--- | :--- |
| 3D | Three-dimensional |
| A/D | Analog to digital |
| AOA | Angle of arrival |
| AWGN | Added Gaussian white noise |
| BLUE | Best linear unbiased estimation |
| BO | Bearings only |
| BPSK | Binary phase-shift keying |
| CAF | Cross-ambiguity function |
| CEP | Circular error probability |
| CIS | Conventional inertial system |
| CM | Combined method |
| COMINT | Communication intelligence |
| CRLB | Cramér-Rao lower bound |
| CTP | Conventional terrestrial pole |
| CTS | Conventional terrestrial system |
| DF | Direction finding |
| DFT | Discrete Fourier transform |
| DOA | Direction of arrival |
| DRC | Doppler rate of changing |
| DSP | Digital signal processor |
| EA | Evolutionary algorithm |
| ECEF | Earth-center earth-fixed |
| ECI | Earth centered inertial |
| EEP | Elliptical error probable |
| EEP | Error ellipse probability |
| EHF | Extremely high frequency |
| EKF | Extended Kalman filter |
| ELINT | Electronic intelligence |
| ER | Electronic reconnaissance |
| ERS | Electronic reconnaissance system |
| EW | Electronic warfare |
| FDOA | Frequency difference of arrival |
| FFT | Fast Fourier transform |
|  |  |


| FM | Frequency modulation |
| :--- | :--- |
| FOA | Frequency of arrival |
| FPGA | Field-programmable gate array |
| GDOP | Geometric dilution of precision |
| GEO | Geostationary orbit |
| GIS | Geographical information system |
| GPS | Global positioning system |
| HOT | Higher order terms |
| HEO | High earth orbit |
| i.i.d. | Independent and identically distributed |
| IF | Intermediate frequency |
| IFF | Identifying friend or foe |
| INS | Inertial navigation system |
| ISE | Initial state error |
| LBI | Long baseline interferometer |
| LEO | Low earth orbit |
| LFM | Linear frequency modulation |
| LNA | Low noise amplifier |
| LO | Local oscillator |
| LOP | Line of position |
| LOS | Line of sight |
| LPF | Lowpass filter |
| LS | Least squares |
| MEO | Medium earth orbit |
| MGEKF | Modified gain extended Kalman filter |
| ML | Maximum likelihood |
| MLE | Maximum likelihood estimation |
| MSE | Mean square error |
| MUSIC | Multiple signal classification |
| NED | North-east-down |
| NLS | Nonlinear least squares |
| NULA | Nonuniform linear array |
| PF | Particle filter |
| PRC | Phase rate of changing |
| PRF | Pulse repetition frequency |
| PRI | Pulse repetition interval |
| PSK | Phase shift keying |
| PSO | Particle swarm optimization |
| RAAN | Right ascension of the ascending node |
| RF | Radio frequency |
| RMS | Root mean square |
| RMSE | Root mean square error |
| SEP | Spherical error probable |
| Space electronic reconnaissance |  |
| Signal Intelligence |  |
| Signal-to-noise ratio |  |
|  |  |


| STD | Standard error |
| :--- | :--- |
| STK | Satellite tool kit |
| SVD | Singular value decomposition |
| TDOA | Time difference of arrival |
| TDRSS | Tracking and data relay satellite system |
| TLS | Total least squares |
| TOA | Time of arrival |
| TTC\&M | Telemetry, tracking, command, and monitoring |
| UAV | Unmanned aerial vehicle |
| UHF | Ultra high frequency |
| UKF | Unscented Kalman filter |
| ULA | Uniform linear array |
| VF | Video frequency |
| VHF | Very high frequency |
| WGS | World geodetic system |
| WLS | Weighted least squares |

## 1

## Introduction to Space Electronic Reconnaissance Geolocation

### 1.1 Introduction

With the rapid development of aerospace technology, space has gradually become the strategic commanding point for defending national security and providing benefits. As the electronic reconnaissance satellite is able to acquire the full-time, all-weather, large-area, detailed, near real-time battlefield information (such as force deployment, military equipment, and operation information), it has become a powerful way to acquire information and plays an important role in ensuring information superiority [1, 2]. In the early 1960s, the United States launched the first general electronic reconnaissance satellites in the world - Grab and Poppy - to collect electronic intelligence (ELINT) on Soviet air defense radar signals. Intelligence from Grab and Poppy provided the location and capabilities of Soviet radar sites and ocean surveillance information to the US Navy and for use by the US Air Force. This effort provided significant ELINT support to US forces throughout the war in Vietnam [3].
Space electronic reconnaissance (SER) refers to the process in which signals from various electromagnetic transmitters are intercepted with the help of man-made satellites, and then features of signal are analyzed, contents of signal are extracted, and the position of transmitters are located [1-5]. The main tasks for space reconnaissance includes: intercepting signals from various transmitting sources such as radars, communication devices, navigation beacons, and identification friend or foe (IFF) transponders, determining the tactical or technological parameters and location, and identifying its type, purpose, and the related air defense system and weapon system; intercepting and analyzing signals of remote control and telemetry and estimating its weapon system performance, experimental situations and development trend; intercepting and monitoring radio communications, analyzing the signal features and determining the location of the transmitters, interpreting and deciphering the communication contents from which the potential military actions and operation plans can be perceived; long-term
monitoring of the changes in the electromagnetic transmitters and obtaining the information such as electronic equipment development status and rules of force deployment and activities.
According to the intended purpose, the application of the SER system can be classified into radio frequency spectrum surveillance, ELINT, communication intelligence (COMINT), signal intelligence (SIGINT), battlefield surveillance, and characteristic measurement intelligence reconnaissance. The major reconnaissance objects are transmitters from air, space, land, and sea. The major reconnaissance signal types include radio signals, short wave and ultra-short communication signals, satellite communication signals, microwave and troposcatter communication signals, data link signals, IFF signals, navigation signals, and space data link signals. The band of the reconnaissance objects ranges from short wave, ultra-short wave, VHF (very high frequency), UHF (ultra-high frequency, L band, S band, C band, X band, Ku band, Ka band to EHF (extremely high frequency) band, while the frequency can range from 0.3 MHz to 70 GHz .

Generally speaking, SER tasks are mainly conducted by electronic reconnaissance satellites on a low earth orbit (LEO) (which includes a sun synchronous orbit, polar orbit, the orbit with the inclination near the critical value, and an inclined orbit) and electronic reconnaissance satellites on a medium earth orbit (MEO) and a high earth orbit (HEO) (highly elliptical orbit or geostationary orbit), and a near-SER (vehicle in the stratosphere or a suborbital vehicle). The altitude of the electronic reconnaissance satellites on a low orbit is relatively low, most often $300-1100 \mathrm{~km}$ with an inclination greater than $50^{\circ}$. Thus a relatively accurate location for the transmitters can be achieved. These satellites can also be applied to monitor the emitters on the sea through the reconnaissance and location of the radar or communication signal on vessels. The reconnaissance can be run with one satellite or a multiple-satellite network. Typical reconnaissance systems are the US Semos- $F$ series electronic reconnaissance satellite, the US White Cloud series electronic ocean surveillance satellite and the former USSR Tselina series electronic reconnaissance satellite. The orbit altitude of a synchronous orbit reconnaissance satellite is generally about 36000 km . The significant advantages are its wide coverage, stability over earth and all-weather, $7 / 24$ continuous reconnaissance, and monitor of the electromagnetic signals from one particular area on earth. A typical HEO reconnaissance system is the US Magnum series electronic reconnaissance satellite. A highly elliptical orbit electronic reconnaissance satellite is primarily used for the continuous reconnaissance and monitoring of the areas with high altitude. It can make up for the disadvantages of poor reconnaissance performance of a synchronous orbit electronic reconnaissance satellite in such areas. A typical system is the US Jump Seat series electronic reconnaissance satellite.
As position information is one of the most important parts in the intelligence generated from the electronic reconnaissance (ER) system, location technology plays a crucial role in the SER and determines the means of operation for the entire reconnaissance satellite. This book introduces various concepts, theories, and methods on electronic reconnaissance geolocation in great detail, and discusses the direction-finding geolocation, geolocation based on TDOA (time difference of arrival), geolocation based on TDOA-FDOA (frequency difference of arrival), geolocation by a single satellite based on kinematics, and geolocation based on a near-space platform, and at the same time analyzes and explores in depth the orbit determination of a satellite using direction finding and frequency information from a space platform.

### 1.2 An Overview of Space Electronic Reconnaissance Geolocation Technology

According to the space location of the transmitters, two types of signals can be intercepted from space: the transmitters on the earth's surface (land, sea, and air) and the satellite transmitters in space. According to different observation platforms, the electronic reconnaissance system can be based on a satellite or a near-space platform. Therefore, three reconnaissance geolocation technologies are discussed here, that is, geolocation of a ground emitter, geolocation of a space emitter, and geolocation using a near-space platform.

### 1.2.1 Geolocation of an Emitter on the Earth

Through the geolocation of the transmitters of the earth's surface, information from various radars, wireless communication stations, and navigation stations can be revealed, which is rather meaningful and valuable for the military. The earth's surface here is in a broad sense that covers land, sea, and lake surface and low altitude air.
The fundamental characteristic of SER geolocation is to locate the satellite through the intersection between the geolocation line and the a priori information of the ground emitter on the earth's surface, as shown in Figure 1.1. According to the number of electronic reconnaissance satellites, the geolocation method can be classified into the geolocation method by a single satellite, the geolocation method by dual-satellite, and the geolocation method by multiple satellites.

### 1.2.1.1 Geolocation Method by a Single Satellite

Naturally, using this method, the geolocation can be done using one satellite. This method can be further classified into:

1. The geolocation method by a single satellite based on the line of sight (LOS).

This is a traditional and widely used method [6-10], which locates the transmitters through the intersection between the oriented LOS generated from the two dimensional (2D) direction finding system on the satellite with the earth's surface, as shown in Figure 1.2. The advantage of this method is that it can realize 'instantaneous' geolocation, sometimes called a 'single-pulse geolocation'. However, as the LOS in a 3D space must be determined with the 2D direction finding antenna array or a multiple beam antenna, large numbers of antennas and receivers are, generally speaking, required. In addition, the attitude measurements of the observing satellite, including yaw, pitch, and roll angle, also need to be accurate enough.
2. The geolocation method by a single satellite based on particle kinematic parameters.

This novel method locates the transmitters by the rate of changing information, such as the frequency of the received signal and/or the time of arrival (TOA), or the phase rate of changing information of a long baseline interferometer (LBI) over a period of time. It
draws the relative moving information from kinematic features of the transmitters against the satellite observatory platform. Featured with a payload of small volume, light weight, and low power consumption, it is suitable for a microsatellite or a nanosatellite.

According to the orbit altitude, the geolocation by a single satellite based on the LOS can be classified into three types: geolocation by a single LEO satellite, geolocation by a single HEO satellite, and geolocation by a single satellite on a highly elliptical orbit.


Figure 1.1 Geolocation of an emitter on the earth's surface by a single LEO reconnaissance satellite


Figure 1.2 The geolocation method used by a single satellite based on the line of sight (LOS)

1. The geolocation by a single LEO satellite

As the LEO satellite's orbit is low, generally $500-1100 \mathrm{~km}$ with an inclination of more than $50^{\circ}$, it is closer to the transmitters on earth compared with the HEO satellites, the signals intercepted by satellites are stronger, and the position can be estimated quite accurately. In addition, its cost is low for production and launching as smaller, low-gain antennas can be applied. Therefore, this was one of first systems developed in history, such as the first ELINT satellites - Grab and Poppy satellite series of the United States. These satellites can also be applied to monitor the targets on the sea surface through the reconnaissance and geolocation of the radar or communication signal on vessels. The reconnaissance can be run with one satellite or a multiple satellite network. Typical reconnaissance systems are the US Semos-F series electronic reconnaissance satellite, the US White Cloud series electronic ocean surveillance satellite, and the former USSR Tselina series electronic reconnaissance satellite. Its disadvantage is that as the LEO satellite's orbit is low the reconnaissance of the same place cannot be kept for long and the instantaneous coverage field of the satellite may be narrow compared with other orbits.
2. The geolocation method of a single HEO satellite [1-5]

As the satellite is far away from the transmitters, for example, the altitude of the earth's stationary orbit is approximately 35800 km , the advantage is that the coverage area is very wide. It can remain stationary over the earth and all-weather, full-time (7/24) monitoring over one area, especially the hotspot ones, can be conducted. In comparison with the LEO satellite, there is no orbit revisit period problem for HEO satellites. They can transmit the reconnaissance data downwards to a ground station for real-time support of tactical operations, which is very meaningful for military strategy and tactics. A typical reconnaissance system is the US Magnum series electronic reconnaissance satellite. However, as the emitter on earth is far away from the satellite, the intercepted signals are quite weak. It is necessary to intercept the signals with a large-diameter, high-gain antenna. As a result of the long distance, there is a high demand placed upon direction finding accuracy for geolocation of the emitter on earth, which makes it quite challenging as far as technology is concerned.
3. The geolocation method by a single satellite on a highly elliptical orbit based on direction finding

The apogee of such a satellite is approximately 38720 km and perigee is about 400 km with inclination of $63.4^{\circ}$. The satellite is primarily used for the continuous reconnaissance and monitoring of the areas with high altitude. It can conquer the disadvantages of poor reconnaissance performance of a synchronous orbit electronic reconnaissance satellite in such areas. Aypical system is the US Jump Seat series electronic reconnaissance satellite.

### 1.2.1.2 The Geolocation Method Based on TDOA-FDOA of a Dual Satellite

Naturally, with this method, dual satellites are formed in one group for reconnaissance. If the signals from the same transmitters can be intercepted by dual satellites at the same time, the geolocation of the emitter on earth could be achieved after signal processing and location estimation [1-5, 11-16].
Generally speaking, in such a system, two satellites may cooperate with each other and are on a same orbit to measure the signal TDOA and FDOA parameter. The basic principle is: as the distance between the ground emitter and two satellites are different, the signal arrives at two satellites at different times. If the TDOA can be calculated, a hyperboloid of revolution with


Figure 1.3 The geolocation method by dual satellites
two satellites as the focus can be determined. Additionally, as speed, that is, radial velocity of two satellites, makes a different projection on the line between the transmitter and reconnaissance satellite, the Doppler frequency shift of the intercepted signals will not be the same. Meanwhile, measurement of the FDOA can be applied to determine a constant Doppler difference curve of revolution with two satellites as the focus. Making an intersection between the hyperboloid of revolution based on equal TDOA and equal FDOA curves of revolution, we can get a circle whose axis is the line of the satellites. Then if we make an intersection between the circle and the earth's surface, we will get two positions. Delete the ambiguous one and we can locate the transmitters. This is shown in Figure 1.3.
Because the LEO satellites move fast and the Doppler frequency difference between two satellites is significant, this method is featured as geolocation by short time accumulation and high accuracy for an LEO dual-satellite reconnaissance system. If dual satellites are on the same orbit, the distribution of geolocation is irrelevant with latitude compared with the three-satellite TDOA geolocation system. As the geolocation by the TDOA and FDOA is on the basis of a cross-ambiguity function (CAF) over the length of a certain duration of signals, the geolocation of the multiple signals can also be realized, even if they are at the same frequency and transmitted at a same time. The disadvantage of this method is that there is the poor geolocation accuracy area near to the subsatellite track. It is a result of the geolocation circle formed by the TDOA and FDOA plane and the earth's surface being nearly tangential in the area close to the subsatellite track. In addition, in order to measure the TDOA and FDOA from received signals in the process of geolocation, the correlation between the two signals must be computed and the signal or signal data should be sent to the same place, such as the primary satellite, ground station, or the tracking and data relay satellite system (TDRSS). Thus, it exerts a high demand upon the high-speed satellite-to-satellite data link or satellite-to-land data link, and the synchronization of time and frequency between the satellites. Besides, left or
right ambiguity is another issue that should be deal with - it is hard to judge whether the target transmitters are located on the left or the right of a subsatellite track in mathematics, that is, the ambiguous geolocation problem. We have to resort to other technical measures such as direction finding information to solve such a problem. Therefore, technically speaking, it is more complicated to realize the dual-satellite TDOA-FDOA geolocation system compared with a single satellite based on the LOS and geolocation by multiple satellites based on TDOA.

### 1.2.1.3 Geolocation Method by Multiple Satellites Based on TDOA

This system locates the transmitters by a set of $N$ satellites ( $N \geq 3$ ). Among the multiple satellite geolocation system based on TDOA, the three-satellite geolocation based on TDOA has been discussed in many papers $[1-5,11,17-21]$.
This method locates the emitters on the earth's surface by TDOAs of the signals arriving at multiple satellites. The basic theory is that as distances between the emitters on earth and different satellites are different, the times of the transmitted signals arriving at the transmitters are also different. So the TDOA of the intercepted signals at any two satellites from the transmitters can determine a hyperboloid of revolution with two satellites as the focus point. The TDOA of the signals arriving at another two of the satellites from the transmitters can determine another hyperboloid of revolution, with two satellites as the focus. Thus the two hyperboloids of revolution can be intersected by each other to form a curve, which further intersects with the earth's surface to obtain two points, which are generally located at the two sides of the earth. When the ambiguous point is ascertained (because normally the ambiguous point is in the other hemisphere of the earth and the transmitting source cannot be at such a point from a priori knowledge), the position of the transmitters on the earth's surface can be located. This process is shown in Figure 1.4.
In the geolocation process of the transmitters, we need the a priori information of the transmitters on the earth's surface so that the geolocation is influenced by the altitude error of


Figure 1.4 Geolocation by three satellites based on TDOA
the transmitters. However, as the altitude of the emitter on an ocean is always approximately zero, the acceptable geolocation accuracy can be achieved. Using this method, surveillance over the transmitters on an ocean can be realized in a very large area ( $3000-7000 \mathrm{~km}$ diameter). An LEO constellation by three to four satellites based on the TDOA is generally formed. For example, in typical TDOA satellites - US first and second generation White Cloud series electronic ocean surveillance satellites - the orbit altitude is approximately 1100 km and inclination is about $63^{\circ}$.
This method is featured with relatively simple, high accuracy in large areas as well as the capability of instantaneous geolocation (sometimes called 'single-pulse geolocation'). In the coverage area of multiple satellites, the highest accuracy is near the subsatellite point, which makes it very suitable for the geolocation of strong power pulsing signal sources like radar. The disadvantage is the changing geolocation accuracy within certain ranges and poor performance when three satellites are in an approximate straight line due to the unstable satellite geometry, because satellites are generally not on the same orbit. In addition, there are other problems such as the 'common view' problem of three satellites, which means that multiple satellites should intercept the same pulse, the TDOA ambiguity problem of a high pulse repetition frequency (PRF) radar signal would occur, the matching problem of pulse and the synchronization problem of the time between satellites would interfere, and so on.

### 1.2.2 Tracking of an Emitter on a Satellite

The geolocation of a satellite emitter refers to tracking of the emitter satellite through the geolocation of its signals by another reconnaissance satellite. As the satellite moves according to certain orbit rules in space, we also call this process an orbit determination process. Its application is wide and it can be classified into two groups: the first application is for the spacecraft space telemetry, tracking, command, and monitoring (TTC\&M), which aims at tracking a cooperative satellite; the second application aims at tracking uncooperative satellites through which the electromagnetic surveillance of the space emitters can be achieved and the information of the orbit, status, and function of satellites can be deduced. In this book we will focus on the second group and take the issue of orbit determination of a moving satellite transmitter by a single satellite into consideration.
According to the type of signal, it can be classified into two groups. The first one is the LOS geolocation system, or bearings only tracking system, which is measured by photoelectric sensors such as a camera or infrared imaging sensor. The tracking of the emitter of a satellite can be made through measurement of the emitter's LOS information. The second one is the passive tracking system which intercepts the radio signal transmitted from the emitter. With some parameter estimation and tracking algorithms, the location, speed, or elements of the uncooperative satellite orbit can be identified. The signals intercepted from the satellite are mostly the satellite-to-satellite data-link signals, such as communication, command and control, navigation or telemetry signals. In this way uncooperative orbit determination and tracking can also be achieved.
As shown in Figure 1.5, when signals are sent from the target satellite to the geostationary data relay satellite, such as the TDRSS, with the analysis and parameter estimation of the signals intercepted by a reconnaissance satellite, the location, speed, and elements of orbits of the uncooperative satellite can be estimated.


Figure 1.5 Satellite-to-satellite geolocation

With the passive working mode on photoelectric sensors (such as a camera or infrared imaging sensor), only the target's bearings information can be obtained. The bearings or LOS can be measured accurately with an accuracy of approximately $4^{\prime \prime}-10^{\prime \prime}$ in some applications. On the other hand, after passively intercepting radio signals, the target's bearing, frequency, and its changing rate may also be measured. The accuracy of these parameters is related to the signal processing algorithms and device technologies. Generally speaking, the latter's accuracy is lower than the accuracy achieved with the former one.

### 1.2.3 Geolocation by Near-Space Platforms

Near space refers to the space between the highest altitude at which a contemporary airplane can fly and the lowest altitude at which a satellite can fly [22-24]. Currently speaking, aircraft in the near space include a free balloon, airship, unmanned aerial vehicle (UAV), and hypersonic UAV. As the full-time, large-scaled and all-weather electronic surveillance can be conducted over one area, research of this geolocation application is rather useful. The geolocation by a near-space platform can be classified into the geolocation by multiple near-space platforms, such as the geolocation based on TDOA and the geolocation by a single near-space platform based on particle kinematics.

Based on the ground and space emitter, this book introduces various geolocation methods and their theories, methods, analysis, and technologies. We hope it will be beneficial to your understanding of the SER geolocation theory and methods.

### 1.3 Structure of a Typical SER System

Normally, the electronic reconnaissance satellite consists of the satellite platform and the effective payload. As the carrier of the effective payload, the satellite platform is composed of TTC\&M equipment, power, and shell. With reference to the SER satellite, the effective payload refers to the electronic reconnaissance equipment. Its primarily purpose is to intercept, analyze, and store the electromagnetic transmitter signals of the electronic equipment. The


Figure 1.6 Components of the electronic reconnaissance effective payload
effective payload consists of an antenna, receiver, signal processor, system administration and control equipment, and storage and transmittal equipment. For example, the typical structure of an LEO electronic reconnaissance satellite is shown in Figure 1.6.
The functional parts of the effective payload in an SER satellite are listed as follows [25, 26]:

- Reconnaissance antenna

It receives the intercepted electromagnetic signals with some gain, including radar, communication, and telemetry signals. The type of antenna varies according to the task of the satellite. The narrow-beam scanning antenna, multibeam antenna, phase/amplitude comparison antenna, or phased-array antenna can be used.

- Electronic reconnaissance receiver (ELINT receiver)

The electronic reconnaissance receiver is also called the electronic intelligence receiver. It is used to magnify, control, and filter the signals in order to extract particular signals. As a result of the high altitude and wide coverage area of the satellite, it is facing a complicated electromagnetic signal environment. In order to intercept different sources simultaneously, the receiver needs to have a wide frequency coverage, high interception probability, high sensitivity, high accuracy, and strong adaptability of different unknown signals.

- Signal processor

It processes and analyzes parameters the intercepted analog signals from the receiver by changing them into digital signals. As the satellite equipment is strictly limited in volume, weight, and power consumption, the receiver generally performs some simple operations and compares them with those of ground station signal processing, primarily involving recording, storage, or direct transmission.

- Storage and relay equipment

It stores the signal data and processing results collected on the satellite temporarily and transmits the data to the ground stations when the satellite is near ground stations. It can be classified into satellite-to-ground and satellite-to-satellite types. The former's equipment is used to send the information at the terminal to the satellite earth station, but by the latter the reconnaissance data are transmitted to the ground through the geostationary data relay satellite by the satellite-to-satellite link.

- System control and administration equipment

It receives directions and commands from the ground and controls the system to operate according to the task mode set by the ground.

## References

1. Wang, Y. and Liu, Y. (2003) Military Satellite and Application Concept. Beijing: National Defence Industry Press, May 2003 (in Chinese).
2. Lu, Y., Wang, Y., Wu, Z., et al. (2009) Space Information Confrontation. Beijing: National Defence Industry Press, January 2009 (in Chinese).
3. Bernard, R.L. (2009) Electronic Intelligence (ELINT) at NSA, Center for Cryptologic History National Security Agency.
4. Feng, R. (2000) Development of electronic reconnaissance satellites in America. Space International, 6-8 (in Chinese).
5. Chen, M. and Xu, J. (2000) Application and development of modern electronic reconnaissance satellites. Space International, 16-18 (in Chinese).
6. Zhang, R. (1998) The Kinetics and Control of Attitude for Satellite Orbit. Beijing: Beihang University Press, May 1998 (in Chinese).
7. Liu, H., Liu, Z., Jiang, W., et al. (2009) Fusion algorithm for emitter localization based on satellite carried direction finding system. System Engineering and Electronics, 12(31): 2785-2789 (in Chinese).
8. Gong, W., Feng, D., Xie, K., et al. (2005) Research on the self-calibration approach of satellite carried direction finding localization bias. Aerospace Electronic Warfare, 2(21): 25-30 (in Chinese).
9. Yang, B., Zhang, M., and Li, L. (2009) Algorithm for single-satellite DOA localization based on model WGS284. Aerospace Electronic Warfare, 25: 24-26 (in Chinese).
10. Yuan, X. (2005) Research on ground target direction finding localization by remote-sensing satellite. Space Electronic Technology, 1-9 (in Chinese).
11. Ho, K.C. and Chan, Y.T. (1997) Geolocation of a known altitude object from TDOA and FDOA measurements. IEEE Transactions on Aerospace and Electronic Systems, 33(3): 770-783.
12. Pattison, T. and Chou, S. I. (2000) Sensitivity analysis of dual-satellite geolocation. IEEE Transactions on Aerospace and Electronic Systems, 36(1): 56-71.
13. Guo, F. (2008) Analysis of dual-satellites TDOA-FDOA passive localization. Aerospace Electronic Warfare, 22: 20-23 (in Chinese).
14. Guo, F. and Fan, Y. (2008) Combined dual-satellite TDOA and FDOA localization and its error analysis. Journal of Astronautics, 29(4): 1381-1386 (in Chinese).
15. Huang, Z. and Lu, J. (2003) Space-based passive localization and modern small satellite technology. Journal of the Academy of Equipment Command and Technology, 14: 24-29.
16. Qu, W., Ye, S., and Sun, Z. (2005) Algorithm of location iteration for satellite interference location. Journal of Electronics and Information Technology, 27(5): 797-800 (in Chinese).
17. Foy, W. H. (1976) Position-location solutions by Taylor-series estimation. IEEE Transactions on Aerospace and Electronic Systems, AES-12(2): 187-194.
18. Torrieri, D. J. (1984) Statistical theory of passive location systems. IEEE Transactions on Aerospace and Electronic Systems, AES-20(2): 183-198.
19. Zhong, D. (2002) Research on Three-Satellite TDOA Geolocation Based on WGS-84. Changsha: Graduate School of National University of Defense Technology, November 2002 (in Chinese).
20. Zhong, D., Deng, X., and Zhou, Y. (2003) A location method based on WGS-84 earth model using satellite TDOA measurements. PhD Dissertation (in Chinese).
21. Li, J. (2004) Research on four-satellite TDOA location algorithm. Electronic Information Warfare Technology, 7(19): 3-7 (in Chinese).
22. Wang, Y., Li, D., and An, Y. (2009) Application of near-space vehicle in electronic reconnaissance. Aerospace Electronic Warfare, 4(25): 18-24 (in Chinese).
23. Yi, Z. and Li, Q. (2006) Analysis of near-space vehicle and its military application. Journal of the Academy of Equipment Command and Technology, 5(17): 64-69 (in Chinese).
24. Li, Y., Yao, W., Zheng, W., et al. (2006) Type and characteristics of near-space system. Satellite Application, 3(14): 1-6 (in Chinese).
25. Hou, Y., An, W., Guo, F., et al. (2009) Principles of Electronic Warfare. Beijing: Publishing House of Electronics Industry, September 2009 (in Chinese).
26. Leopold, C. (1989) Space-Based Radar Handbook, Norwood, MA: Artech House.

## 2

## Fundamentals of Satellite Orbit and Geolocation

### 2.1 An Introduction to the Satellite and Its Orbit

An artificial earth satellite, also called an artificial satellite, or satellite for short, is an unmanned spacecraft that orbits earth at least once. Since the satellite moves in a regular and cyclic manner around the earth, the fixed, repeated movement path in the inertial coordinate system of the satellite is called the orbit.

### 2.1.1 Kepler's Three Laws

The revolution of the satellite around the earth is very similar to that of planets around the Sun, and its motion basically conforms to Kepler's Laws. If only the 'two bodies' of the spacecraft and central body of the earth are considered without taking into account any effects of other factors, that is, assuming that the earth is a uniform sphere and the satellite is a particle, the motion of the satellite can be considered to be a kind of two-body motion. This two-body motion conforms to the restrictions of Kepler's Laws [1-4].

### 2.1.1. 1 Kepler's First Law

The satellite moves in a quadratic curve with the earth as one focus of the curve. In a polar coordinate system, the satellite motion can be described as

$$
\begin{equation*}
r=\frac{p}{1+e \cos \theta} . \tag{2.1}
\end{equation*}
$$

where $e$ is eccentricity, generally $0 \leq e<1$ for an earth satellite that is making elliptic motion; $p=h^{2} / \mu=a\left(1-e^{2}\right)$ is the semi-latus rectum, where $h$ is momentum moment of the satellite moving relative to the earth center, and $\mu$ is Kepler's constant: $\mu=398600.4418 \mathrm{~km}^{3} / \mathrm{s}^{2}$.
As shown in Figure 2.1, the point most distant from the center of the earth in the orbit of a satellite is called the apogee; the point nearest the center of the earth is the perigee. In general,


Figure 2.1 Diagram of a satellite orbit
$h_{\max }$ and $h_{\min }$, the heights of the apogee and perigee of the satellite above the ground, are given first for the design of the satellite orbit, based on which other parameters of the elliptical orbit can be determined:

$$
\begin{align*}
& \text { Semi-major axis of ellipse : } \quad a=\frac{h_{\max }+h_{\min }}{2}+R_{e} \text {, }  \tag{2.2}\\
& \text { Focal length of ellipse }: \quad c=a e,  \tag{2.3}\\
& \text { Semi - minor axis of ellipse : } \quad b=a \sqrt{1-e^{2}},  \tag{2.4}\\
& \text { Eccentricity }: \quad e=\frac{c}{a}, \tag{2.5}
\end{align*}
$$

where $R_{e}$ is the earth radius.

### 2.1.1.2 Kepler's Second Law

In equal periods of time, the area swept out by the line from the earth to the satellite will be the same when the satellite moves on the orbit. According to this law, instantaneous velocity of the satellite with any radius vector $\mathbf{r}$ can be obtained:

$$
\begin{equation*}
v(\mathbf{r})=\sqrt{\mu\left(\frac{2}{r}-\frac{1}{a}\right)} \tag{2.6}
\end{equation*}
$$

where $a$ is the semi-major axis of orbital ellipse and $r=\|\mathbf{r}\|$ is distance scalar of the radius vector $\mathbf{r}$. This law reflects the proportional relationship between the speed of the satellite and its location on the orbit: the speed of the satellite in the orbit decreases as its distance from the earth increases.

### 2.1.1.3 Kepler's Third Law

The square of the period $T$ of the satellite to complete one cycle around the earth is directly proportional to the cube of the semi-major axis of the orbital ellipse $a$ :

$$
\begin{equation*}
T^{2}=\frac{4 \pi^{2}}{\mu} a^{3} \tag{2.7}
\end{equation*}
$$



Figure 2.2 Orbits with different eccentricities

This relationship is essentially based on the fact that the force of gravity $G$ to the satellite is inversely proportional to the square of the distance $r^{2}$ between the satellite and the earth center. This force of gravity is center gravity, also called two-body gravity. The solution of the two-body motion equation fully conforms to the Kepler's three laws. Thus a two-body orbit is also called a Keplerian orbit.

### 2.1.2 Classification of Satellite Orbits

Common satellite orbits can be classified according to four aspects, that is, eccentricity, inclination, altitude, and relation to the earth revolution [1, 3, 4].

### 2.1.2.1 Classification by Orbit Eccentricity

According to the shape, the satellite orbit can be classified as a circular orbit and an elliptical orbit. As shown in Figure 2.2, the orbit with eccentricity $e=0$ is a circular orbit; the orbit with eccentricity $0<e<1$ is an elliptical orbit. If $e$ is close to 1 , the orbit is a highly elliptical orbit (HEO).

### 2.1.2.2 Classification by Orbit Inclination

According to different inclinations $i$ between the satellite orbital plane and the earth equatorial plane, the orbit can be classified as an equatorial orbit with $i=0^{\circ}$ (e.g., geostationary orbit (GEO)), polar orbit with $i=90^{\circ}$, and other inclined orbits with different inclinations.
According to the relation between the direction of satellite motion and its rotation direction, an orbit with an inclination of $0-90^{\circ}$ is called a prograde orbit. Conversely, if the inclination is $90-180^{\circ}$, the orbit is called a retrograde orbit. Most satellites are launched into a prograde orbit. In such a case, the earth's rotational velocity will provide a part of the orbital velocity, thus saving energy required by the satellite launch.

### 2.1.2.3 Classifications by Orbital Altitude

According to the orbital altitude, generally the satellite orbit can be classified into GEO, medium earth orbit (MEO), and low earth orbit (LEO).

## 1. Geostationary orbit (GEO)

The satellite in a GEO has an orbital period $T$ equal to the earth's rotational period, and the orbit is a circular orbit in the earth's equatorial plane, $e=0, i=0$. Thus, the GEO is an orbit 35786.04 km above the earth's equator. To ground observers, satellites in such an orbit appear stationary in the sky at their respective locations. Making full use of this relatively stationary feature can simplify satellite navigation, communication, and reconnaissance systems. Three to four satellites are enough to provide global coverage. The GEO is themost used satellite orbit so far.
2. Medium earth orbit (MEO)

An orbit ranging in altitude from 10000 to 15000 km is a MEO. In general, $10-15$ satellites in an MEO can provide global coverage.
3. Low earth orbit (LEO)

In general, an orbit with an altitude below 1500 km is a LEO. More than 40 satellites in such an orbit can cover the whole earth.

A satellite in a HEO has a much longer working life than that in an LEO, since it has less atmospheric effects. Moreover, a satellite in a HEO has a wider coverage range than that in an LEO. However, due to some specific demands of satellite missions (e.g., satellite imaging reconnaissance), satellites prefer to work in the LEO.
The minimum altitude at which, in general, a satellite can maintain free flight is a critical orbital altitude. This altitude is usually $110-120 \mathrm{~km}$. If the satellite's flight altitude is lower than this critical orbital altitude, the satellite will fall down to earth in the end due to the effect of atmospheric drag. In such a case, with the help of the satellite's power and control system, the effect of atmospheric drag perturbation can be offset. An orbit with this flight altitude is defined as an ultra-low earth orbit.

### 2.1.2.4 Classification by Relationships between Satellite Revolution and Rotation Periods

According to this classification method, satellite orbits can be classified as geosynchronous orbit, GEO, sunsynchronous orbit, recursive orbit, quasi-recursive orbit, and so on.

## 1. Geosynchronous orbit

An orbit in which the speed of a satellite orbiting the earth coincides with the speed of earth rotation is called a geosynchronous orbit. An orbit in which the period of a satellite orbiting the earth is an integral multiple of the earth's rotation period is called a supersynchronous orbit. The synchronous orbit and supersynchronous orbit are designed to synchronize the satellite with the earth.
2. Geostationary orbit

A special case of the geosynchronous satellite is the geostationary satellite. When orbital eccentricity and inclination are close to zero, a satellite in such an orbit appears stationary relative to the earth's location. This type of satellite is called a geostationary satellite. At
present, many international and domestic communication satellites use this method of synchronization. Besides communication satellites, a great number of satellites such as a TV broadcast satellite, early warning satellite, navigation satellite, and weather satellite often use GEOs. A geostationary satellite has the following characteristics:

- Orbital period: 23 hours 56 minutes 4.09 seconds;
- Satellite altitude: 35786.04 km ;
- Satellite velocity: $3.074662 \mathrm{~km} / \mathrm{s}$;
- Orbit inclination: $0^{\circ}$; and
- Orbit eccentricity: $e=0$.

Currently, there are a great number of satellites in the GEO. Due to limited orbital position, in 1979 the World Administrative Radio Conference reduced the minimum spacing between orbital satellites to $2.5^{\circ}$.
3. Sunsynchronous orbit

The included angle between a satellite's orbital plane and equatorial plane usually remains changeless, but will revolve around the earth's rotation axis. An orbit in which the revolution direction of the orbital plane around the earth's rotation axis is the same as that of earth revolution and its angular velocity is equal to the average angular velocity of the earth revolution is called a sunsynchronous orbit. The direction and period of rotation of a sunsynchronous orbital plane are the same as the direction and period of an earth revolution.

The relationship among the semi-major axis, eccentricity, and inclination of the sunsynchronous orbit is

$$
\begin{equation*}
\cos i=-K\left(1-e^{2}\right)^{2} a^{7 / 2} \tag{2.8}
\end{equation*}
$$

where $K>0$ is a constant.
From expression (2.8) it can be seen that the inclination $i$ of the sunsynchronous orbit must be more than $90^{\circ}$. In general, the inclination is between $90^{\circ}$ and $100^{\circ}$, that is to say, the sunsynchronous orbit is usually a retrograde inclined orbit and the orbital altitude is $500-1000 \mathrm{~km}$. When the orbit inclination reaches the maximum degree, $180^{\circ}$, it is not difficult to realize that the altitude of a circular sunsynchronous orbit will be lower than 6000 km . Orbit inclination will determine satellite altitude. The larger the inclination, the higher is the satellite.

Due to a retrograde orbit, the satellite will be launched against the direction of the earth's rotation. It therefore needs greater thrust from the carrier rocket.

A satellite in a sunsynchronous orbit revolves in such a way that the satellite keeps the same local time, same direction, and same sunlight condition every time it passes over a ground emitter of the same latitude. The included angle between the orbital plane of a sunsynchronous satellite and the sunshine direction will remain changeless during a satellite revolution. Moreover, the sunsynchronous satellite orbit and sun keep a fixed relative orientation, thus helping long-term and stable satellite work using solar energy and satellite work for optical imaging.

Based on the above-mentioned characteristics, a sunsynchronous orbit is very useful for military, remote sensing, weather forecasting, and resources survey satellites.

## 4. Recursive orbit

An orbit in which a subsatellite track coincides regularly is a recursive orbit. If the number of regressions is $N$, the track of the satellite orbiting the earth for $N$ circles in a day just coincides with the original track, that is, regression to the original orbit. If the regression
period of the satellite is not a day, but several days or weeks, this type of orbit is called a quasi-recursive orbit.

If the satellite is required to observe repeatedly a specific area on the earth's surface at a specific time every day, a sunsynchronous recursive orbit is required. Combining the sunsynchronous orbit and the recursive orbit, this type of orbit can be used for repeated observations at the same local means solar time in most areas on the earth's surface on the same latitude at regular intervals. Earth resources survey satellites and reconnaissance satellites often use this orbit.

## 5. Polar orbit

An orbit with an inclination $i=90^{\circ}$ is a polar orbit. Only a satellite in a polar orbit can pass above or nearly above both poles of the earth on each revolution. In engineering, an orbit with an inclination deviating $90^{\circ}$ but in which a satellite can still pass above both poles is also called a polar orbit, for example, a sunsynchronous orbit. In orbit design, a polar orbit is often designed for global coverage. Satellites in a polar orbit such as a weather satellite, photographic reconnaissance satellite, and remote-sensing resources satellite can 'see' the whole earth surface including both poles.

### 2.2 Orbit Parameters and State of Satellite

### 2.2.1 Orbit Elements of a Satellite

From theoretical mechanics we can see that, if we take the earth as a regular sphere with uniformly distributed density, the earth's attraction to a satellite can be equivalent to a particle, that is, all the mass is concentrated on the earth's center. Thus, the earth and the satellite form a simple two-body system. Since the satellite mass $m$ is very small relative to the mass of the earth, the effect of the satellite on gravity is negligible. In such a case, according to the law of universal gravitation, the force of gravity $\boldsymbol{F}$ upon the satellite is $[1,3,4]$

$$
\begin{equation*}
\boldsymbol{F}=-\frac{G M m}{r^{2}} \frac{\boldsymbol{r}}{r}, \tag{2.9}
\end{equation*}
$$

where $G$ is universal gravitational constant, $r$ is the position vector of the satellite in ECI (earth-center inertial) coordinates, and $r$ is the distance from the center of mass of the satellite to the earth center, $M$ is the mass of earth.
According to Newton's Second Law, the satellite motion equation can be expressed as

$$
\begin{equation*}
\ddot{\boldsymbol{r}}=\frac{\boldsymbol{F}}{m}=-\frac{G M}{r^{2}} \frac{\boldsymbol{r}}{r}=-\frac{\mu \boldsymbol{r}}{r^{3}}, \tag{2.10}
\end{equation*}
$$

where $\mu=G M$ is the gravitational constant. According to the latest recommendations of the International Astronomical Union (IAU), the gravitational constant $\mu=3.986004415 \times 10^{14} \mathrm{~m}^{3} / \mathrm{s}^{2}$. In the case where there are no special instructions, this value should be used as the gravitational constant in future simulations. From expression (2.10) we can see that the two-body motion equation is a ternary second-order differential equation. In solutions of this equation, six integral constants are required to be determined to describe the motion of the satellite in the inertial coordinates system. We often call these six integral constants orbit elements.


Figure 2.3 Satellite orbit elements

The above-mentioned six integral constants are usually expressed by the semi-major axis $a$, eccentricity $e$, inclination $i$, right ascension of the ascending node (RAAN) $\Omega$, argument of perigee $\omega$, and mean anomaly $M$ or $M_{0}(\tau)$. They are called orbit elements. Complete descriptions are shown in Figure 2.3 [1, 3-5].
As shown in Figure 2.3, the semi-major axis $a$ and eccentricity $e$ determine the size and shape of an elliptical orbit; inclination $i$ is defined as the angle between the orbital plane and the earth's equatorial plane, which is positive at the ascending node anticlockwise from the equatorial plane, ranging from $0^{\circ}$ to $180^{\circ} ; \operatorname{RAAN} \Omega$ is defined as the geocentric angle of the ascending node and the spring equinox in the ECI coordinates system. There are two intersection points of the satellite orbit and the terrestrial equator. One is the ascending node, a point at which the orbit crosses the equator going from south to north. RAAN is expressed positive anticlockwise from the axis of ECI coordinates pointing to the spring equinox, ranging from $0^{\circ}$ to $360^{\circ}$. Inclination $i$ and RAAN $\Omega$ determine the location of the orbital plane in space. Argument of perigee $\omega$ is defined as the angle between the ascending node in the orbital plane and apse line of the perigee, which is positive from the ascending node in the direction of satellite revolution, ranging from $0^{\circ}$ to $360^{\circ}$. The argument of the perigee determines the azimuth of the ellipse in the orbital plane. The mean anomaly $M$ determines the location of the satellite in the orbital plane. Refer to the definition in Section 2.2.2. The mean anomaly $M$ can also be expressed in $\tau$, whichis the time perigee and is used to describe the starting point of the satellite revolution time. See the Appendix for the transformational relation between orbit elements and the ECI coordinates system.


Figure 2.4 Eccentric argument of perigee $E$ and true perigee $f$

### 2.2.2 Definition of Several Arguments of Perigee and Their Correlations

Since the satellite has a periodic motion in an elliptical orbit, assuming the revolution period $T$ and average angular velocity $n$, one can obtain $[4,5]$

$$
\left.\begin{array}{l}
n=\sqrt{\mu / a^{3}}  \tag{2.11}\\
T=2 \pi \sqrt{a^{3} / \mu}
\end{array}\right\}
$$

that is to say, when the semi-major axis $a$ is known, from expression (2.11) one can obtain the orbit period $T$ and average angular velocity $n$. An auxiliary value $E$ is introduced. This value is called the eccentric argument of perigee, as shown in Figure 2.4.
In Figure 2.4, $\mathrm{O}_{\mathrm{e}}$ is the earth center, a focal point of the elliptical orbit of the satellite, $\mathrm{O}^{\prime}$ is the geometric center of the ellipse, and $f$ is the true argument of the perigee of satellite S . Draw an auxiliary circle with $\mathrm{O}^{\prime}$ as the center of the circle and semi-major axis $a$ as the radius, and draw a vertical line SH passing through point S and perpendicular to the direction of the perigee $\mathrm{O}_{\mathrm{e}} \mathrm{A}$; take the point of intersection of the extension line of line SH and the auxiliary circle as $\mathrm{S}^{\prime}$, and connect points $\mathrm{O}^{\prime}$ and $\mathrm{S}^{\prime}$. The angle between $\mathrm{O}^{\prime} \mathrm{S}^{\prime}$ and $\mathrm{O}^{\prime} \mathrm{A}$ is the eccentric argument of perigee $E$.
From Figure 2.4 it can be seen that the relation between the eccentric argument of perigee $E$ and ellipse parameters is $a e \cos E=a-r$, and thus the elliptic equation expressed in the eccentric argument of perigee $E$ can be obtained as

$$
\begin{equation*}
r=a(1-e \cos E) \tag{2.12}
\end{equation*}
$$

Since $r$ and $E$ are both functions of time $t$, by differentiating and simplifying expression (2.12) one can obtain

$$
\begin{equation*}
E-e \sin E=n(t-\tau) \tag{2.13}
\end{equation*}
$$

where $\tau$ is a new integral constant. When $t=\tau, r$ is minimum, that is, the perigee of the satellite. So $\tau$ is the time past perigee. For the convenience of expression, $M=n(t-\tau)=$ $n t+M_{0}$ is often used, where $M$ is the mean argument of perigee, which represents the angle
between the perigee of an orbit and the position of the satellite orbiting at an average angular velocity $n ; M_{0}$ is the initial mean argument of the perigee or epoch mean argument of perigee, which indicates that the mean argument of perigee of the satellite at $t=0$.
The definitions of several most common anomalies $f, E$, and $M$ and their relationships have been given. The integral constant $\tau$ can be replaced by these three anomalies (in particular $M$ ) as orbit elements.

### 2.3 Definition of Coordinate Systems and Their Transformations

When we talk about geolocation, we shall solve the first question of which coordinates system the location to be localized lies in. In different coordinate systems, the location has different expressions. Moreover, the motion equation is usually described in the inertial coordinate system, while observations of electronic reconnaissance are generally obtained from the satellite measurement body coordinate system. The satellite itself moves or revolves at a high speed. Therefore, we must study how to establish the coordinate systems of geolocation and mutual transformation between coordinate systems.

### 2.3.1 Definition of Coordinate Systems

In space electronic reconnaissance and geolocation, coordinate systems involved include the earth-center inertial (ECI) coordinate system, earth-centered earth-fixed (ECEF) coordinates, geodetic coordinate system, topocentric-horizon coordinate system, and satellite body coordinate system [5, 6], and so on.

### 2.3.1.1 Earth-Center Inertial (ECI) Coordinate System - $\left\{\operatorname{System} c: x_{c}, y_{c}, z_{c}\right\}$

The ECI coordinate system, also called the geocentric mean equatorial coordinate system, geocentric equatorial coordinate system, or epoch geocentric coordinate system, is a type of celestial coordinate system. With the center of the earth as its origin, axes of coordinate systems are connected to the celestial sphere with the center of the earth as the center in a fixed way. Axis $z_{c}$ coincides with the earth's rotation axis, while axes $x_{c}$ and $y_{c}$ are perpendicular to one another and fixed on epoch equatorial plane. Axis $x_{c}$ points at the spring equinox in the earth's orbit around the Sun, while axes $y_{c}, x_{c}$, and $z_{c}$ constitute a right-handed coordinate system.
Since the earth's rotation axis is constantly changing, in order to establish a uniform celestial coordinate system, which is close to an inertial coordinates system, a time $t_{0}$ is selected as a standard epoch and a celestial coordinate system is established based on the mean pole and mean equinox of this epoch. A celestial coordinate system established in such a way, in fact, is an instantaneous mean celestial coordinate system at the $t_{0}$ epoch, also called a conventional inertial system (CIS). The current system used in the aerospace survey and data processing is the J2000.0 geocentric coordinate system, which is defined with the mean equator and mean equinox at the standard epoch of 12:00 1 January 2000, Gregorian calendar. J2000.0 CIS (celestial coordinate system) is defined as a right-handed Cartersian coordinate system, with the center of the earth as the origin, the $z$ axis pointing at J2000.0 mean pole and the $x$ axis pointing at $\mathbf{J} 2000.0$ mean equinox.

If the effects of precession, mutation and polar migration are taken into account, it is necessary to use a multiple-coordinate system to describe accurately the celestial coordinates of one object at one time. Reference [4] has given definitions of several coordinate systems.

### 2.3.1.2 Earth-Centered Earth-Fixed (ECEF) Coordinate System - \{System $e: x_{e}$, $\left.y_{e}, z_{e}\right\}$

The origin of ECEF coordinates is at the center of the earth. The axes of the coordinate system are connected to the earth in a fixed way. Axis $z_{e}$ coincides with the earth's rotation axis and axes $x_{e}$ and $y_{e}$ are perpendicular to one another and fixed on the equatorial plane. Axis $x_{e}$ points outward from the earth's center at the point of intersection of the Greenwich meridian and the equator; axes $y_{e}, x_{e}$, and $z_{e}$ constitute a right-handed coordinate system. Due to the earth's rotation, at one time only the right ascension $\theta$ of one prime meridian differs between ECEF coordinates and the epoch geocentric equatorial coordinate system. Since ECEF coordinates take a fixed location and direction of the earth as the reference, ECEF coordinates of one point on the earth are always fixed.
Figure 2.5 shows a general view of the ECI coordinate system and ECEF coordinates.

### 2.3.1.3 Earth Geodetic Coordinate System - $\{$ System $L, B, H\}$

Sometimes for convenience, such parameters as longitude, latitude, and altitude are used to indicate geographic orientation of point location, that is, point location in geodetic coordinates. The earth geodetic coordinate system is a coordinate system taking the prime meridian plane,


Figure 2.5 Relationship between ECI coordinates and ECEF coordinates
equatorial plane, and spherical surface of the reference ellipsoid as the coordinate planes, that is, geodetic coordinates of one point on the earth are usually expressed in geodetic longitude $L$, geodetic latitude $B$, and geodetic altitude $H$.
Geodetic coordinate parameters of point P are defined as follows. The angle between the prime geodetic meridian plane and another geodetic meridian plane passing through point P is called the geodetic longitude, expressed as $L$; if the point is in the eastern hemisphere, its longitude is called east longitude and if in the western hemisphere, its longitude is called west longitude. The angle from the normal of the point to the equatorial plane constitutes geodetic latitude, expressed as $B$; if the point is in the northern hemisphere, its latitude is called north latitude and if in the southern hemisphere, its latitude is called south latitude. The distance between the point along the normal and the surface of the reference ellipsoid is called geodetic altitude, expressed as $H$. The altitude is calculated from the surface of the reference ellipsoid, positive outwards and negative inwards.
Since the earth is not a regular ellipsoid, the definition of different origins of coordinates and ellipsoidal curvature should correspond to different reference ellipsoid planes to establish different geocentric coordinate systems, for example, the China Geodetic Coordinate System 1980, the Beijing Geodetic Coordinate System 1954, and the WGS-84 Coordinate System of the United States Department of Defense (refer to Section 2.4).
Of all the geodetic coordinate systems, the most common one is the geodetic coordinate system expressed in longitude, latitude, and altitude. It is used in the most common output forms of navigation systems, such as the GPS (global positioning system) and the INS (inertial navigation system. However, in some geolocation calculations, Cartesian coordinates may be more convenient to use. It is necessary to transform the coordinates from one to another.
Due to the effect of polar migration, the direction of the equatorial plane is slowly changing. Reference [4] defines several different geodetic coordinates with polar migration being considered.

### 2.3.1.4 Topocentric-Horizon Coordinates Systems - $\{$ System $g\}$ and $\{$ System $\boldsymbol{n}\}$

The topocentric coordinate system is a coordinate system established with the center of the observer's location as the origin of the coordinate system. It has many similar definitions, for example, the topocentric-horizon coordinate system, north-east-down (NED) coordinate system, launching coordinate system, vertical measuring coordinate system, and normal measuring coordinate system [7].
The most common definition of the topocentric coordinate system is the topocentric-horizon coordinate system, expressed in this book as $\left\{\right.$ System $\left.g: x_{g}, y_{g}, z_{g}\right\}$. This coordinate system takes the topocenter as the origin of coordinates: the $x_{g}$ axis points due east, the $y_{g}$ axis points due north, and the $z_{g}$ axis is perpendicular to the horizon surface of the earth and the point zenith. The $z_{g}$ axis and the other axes constitute a right-handed coordinate system. Since the earth is not a regular sphere, but an irregular ellipsoid, the pointing of the $z_{g}$ axis deviates an angle of $\delta(\delta<4 \mathrm{mrad})$ from the geocentric direction.
Another topocentric coordinate system is the NED coordinate system, expressed as \{System $\left.n: x_{n}, y_{n}, z_{n}\right\}$. This coordinate system takes the topocenter as the origin of coordinates: the $x_{n}$ axis points due north, the $y_{n}$ axis points due east, and the $z_{n}$ axis is perpendicular to the surface of the earth and points downwards. The $z_{n}$ axis and the otherer axes constitute a right-handed coordinate system. Since the earth is an ellipsoid, there is also an angle of declination between the $z_{n}$ axis and the earth center.

Coordinate transformation between the topocentric-horizon coordinate systems, $\{$ System $g$ \} and $\{\operatorname{System} n\}$, can be obtained as

$$
\begin{equation*}
\mathbf{X}_{n}=\mathbf{R}_{n g} \mathbf{X}_{g} \tag{2.14}
\end{equation*}
$$

where

$$
\mathbf{R}_{n g}=\left[\begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{array}\right]
$$

and

$$
\mathbf{R}_{n g}=\mathbf{R}_{g n}=\mathbf{R}_{n g}^{-1}
$$

### 2.3.1.5 Satellite Body Coordinate System - $\left\{\operatorname{System} \boldsymbol{b}: \boldsymbol{x}_{\boldsymbol{b}}, \boldsymbol{y}_{\boldsymbol{b}}, z_{b}\right\}$

The satellite body coordinate system, $\{$ System $b\}$, is shown in Figure 2.6. The origin of the coordinates lies in the center of the platform. The $x_{b}$ axis is the longitudinal axis of the airplane, with the forward direction being the heading of the airplane (direction of the satellite motion); the $y_{b}$ axis lies in the starboard of the fuselage plane, perpendicular to the longitudinal axis; and the $z_{b}$ axis is downwards perpendicular to the fuselage plane. Taking an airplane as an example, the definition of the coordinate system is as shown in Figure 2.6.
In general, three attitude angles are provided by the navigation equipment: platform yaw (also called the heading angle or course angle), pitch and roll are Euler angles [8] from the platform NED coordinate system to the body coordinates. For the convenience of expression and calculation, there is a clear and specific definition of positive and negative values in the measurement of the platform attitude angle in the platform NED coordinates: the yaw angle north by east is positive; the pitch angle in the head-up direction is positive; and the roll angle is positive when the right wing is down. Suppose the platform's yaw angle is $\alpha$, the pitch angle is $\beta$, and roll angle is $\varepsilon$. Figure 2.7 shows all attitude angles.


Figure 2.6 Platform body coordinates system (example of airplane)


Figure 2.7 Diagram of definition of platform attitude angles

### 2.3.2 Transformation between Coordinate Systems

Any coordinate system transformation can be divided into coordinate rotation and coordinate translation. Coordinate translation is relatively simple. Here coordinate rotation is introduced.

### 2.3.2.1 Coordinate Rotation

If a vector in an original coordinate system is expressed as $\mathbf{r}$ and in a new coordinate system after rotation as $\mathbf{r}^{\prime}$, then by rotating the $y z$ plane, $z x$ plane, and $x y$ plane an angle of $\theta$ (counterclockwise is positive) around the $x$ axis, $y$ axis, and $z$ axis, respectively, one can obtain $[4,8,9]$

$$
\begin{align*}
\mathbf{r}^{\prime} & =\mathbf{R}_{x}(\theta) \mathbf{r},  \tag{2.15}\\
\mathbf{r}^{\prime} & =\mathbf{R}_{y}(\theta) \mathbf{r},  \tag{2.16}\\
\mathbf{r}^{\prime} & =\mathbf{R}_{z}(\theta) \mathbf{r}, \tag{2.17}
\end{align*}
$$

where

$$
\begin{align*}
& \mathbf{R}_{x}(\theta)=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{array}\right],  \tag{2.18}\\
& \mathbf{R}_{y}(\theta)=\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right],  \tag{2.19}\\
& \mathbf{R}_{z}(\theta)=\left[\begin{array}{ccc}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right] . \tag{2.20}
\end{align*}
$$

The rotation matrix $\mathbf{R}(\theta)$ has the following characteristics:

$$
\begin{equation*}
\mathbf{R}^{-1}(\theta)=\mathbf{R}^{\mathrm{T}}(\theta)=\mathbf{R}(-\theta) \tag{2.21}
\end{equation*}
$$

Any coordinate can be rotated in a certain order by resolving into $x, y$, and $z$ axes. In such a case, the final rotation matrix is the rotation matrix product. More attention should be made to the noncommutative principle of matrix multiplication. The rotation order should not be commutative. Otherwise a different rotation order would obtain different coordinate rotation effects.

### 2.3.2.2 Transformation Relations between Common Coordinate Systems

1. Transformation from the ECI coordinate system $\{$ System $c\}$ to the ECEF coordinate system \{System e\}

As shown in Figure 2.5, the ECI coordinate system $c\left\{x_{c}, y_{c}, z_{c}\right\}$ rotates in the positive direction a right ascension $\theta$ around the $z_{c}$ axis, and transforms into the ECEF coordinate system $e\left\{x_{e}, y_{e}, z_{e}\right\} . \theta$ is the right ascension of the prime meridian of the earth at a certain time. Due to the rotation of the earth, $\theta$ is changing, with a sidereal day as a period. Suppose that the epoch geocentric equatorial coordinates and ECEF coordinates of a vector are $\mathbf{x}_{c}=$ $\left(x_{c}, y_{c}, z_{c}\right)^{\mathrm{T}}$ and $\mathbf{x}_{e}=\left(x_{e}, y_{e}, z_{e}\right)^{\mathrm{T}}$, respectively; then

$$
\begin{equation*}
\mathbf{x}_{e}=R_{z}(\theta) \mathbf{x}_{c} . \tag{2.22}
\end{equation*}
$$

2. Transformation from the ECEF coordinate system $\{$ System $e\}$ to the topocentric-horizon coordinate system $\{$ System $g$ \}

Three steps are required:
a. Translate the origin of coordinates of the system $e\left\{x_{e}, y_{e}, z_{e}\right\}$ from $O_{e}$ to $O_{g}$.
b. Positively rotate around the $z_{e}$ axis a geodetic longitude $L+\pi / 2$ to make the tangent direction of the latitude line point to the east and then obtain an intermediate coordinate system $e^{\prime}\left\{x_{e}^{\prime}, y_{e}^{\prime}, z_{e}^{\prime}\right\}$.
c. Then rotate the coordinate system $e^{\prime}$ with $\pi / 2-B$ around the $x_{e}^{\prime}$ axis to make $z_{e}^{\prime}$ perpendicular to the surface of the earth and pointing upwards, with $y_{e}^{\prime}$ pointing north. These are the coordinates of the topocentric-horizon coordinate system $g\left\{x_{g}, y_{g}, z_{g}\right\}$.
Suppose that the ECEF coordinates of the topocenter are $\left\{x_{o}, y_{o}, z_{o}\right\}$ and the ECEF coordinates and topocentric-horizon coordinates of the vectors are $\mathbf{x}_{e}=\left(x_{e}, y_{e}, z_{e}\right)^{\mathrm{T}}$ and $\mathbf{x}_{c}=\left(x_{c}, y_{c}, z_{c}\right)^{\mathrm{T}}$, respectively. One can then obtain:

$$
\left[\begin{array}{l}
x_{g}  \tag{2.23}\\
y_{g} \\
z_{g}
\end{array}\right]=\mathbf{R}_{x}\left(\frac{\pi}{2}-B\right) \mathbf{R}_{z}\left(\frac{\pi}{2}+L\right)\left[\begin{array}{l}
x_{e}-x_{o} \\
y_{e}-y_{o} \\
z_{e}-z_{o}
\end{array}\right] .
$$

By substituting Equations (2.18) and (2.20) into Equation (2.2), one can obtain

$$
\left[\begin{array}{l}
x_{g}  \tag{2.24}\\
y_{g} \\
z_{g}
\end{array}\right]=\left[\begin{array}{ccc}
-\sin L & \cos L & 0 \\
-\sin B \cos L & -\sin B \sin L & \cos B \\
\cos B \cos L & \cos B \sin L & \sin B
\end{array}\right]\left[\begin{array}{l}
x_{e}-x_{o} \\
y_{e}-y_{o} \\
z_{e}-z_{o}
\end{array}\right] .
$$

This can be expressed as

$$
\begin{equation*}
\mathbf{x}_{g}=\mathbf{C}_{e}^{g}\left(\mathbf{x}_{e}-\mathbf{u}\right) \tag{2.25}
\end{equation*}
$$

3. Transformation from a topocentric-horizon coordinate system, $\{$ System $g\}$, to ECEF coordinates, $\{$ System $e\}$

If the coordinates are transformed from $\{$ System $g\}$ to $\{$ System $e\}$, it can be seen from expression (2.25) that

$$
\begin{equation*}
\mathbf{x}_{e}=\left(\mathbf{C}_{e}^{g}\right)^{-1} \mathbf{x}_{g}+\mathbf{u}=\left(\mathbf{C}_{e}^{g}\right)^{\mathrm{T}} \mathbf{x}_{g}+\mathbf{u} \tag{2.26}
\end{equation*}
$$

4. Transformation from a NED coordinate system to a body coordinate system

It is obvious that it is only a process of coordinate rotation from platform NED coordinates to platform airplane body coordinates, that is,

$$
\begin{equation*}
\mathbf{r}_{t, b}=\mathbf{R}_{x}^{\mathrm{T}}(\varepsilon) \mathbf{R}_{y}^{\mathrm{T}}(\beta) \mathbf{R}_{z}^{\mathrm{T}}(\alpha) \cdot \mathbf{r}_{t, o} . \tag{2.27}
\end{equation*}
$$

### 2.4 Spherical Model of the Earth for Geolocation

In a wide range of applications of satellite geolocation, it is required to use prior geolocation knowledge that the emitter should be located on the earth's surface. Therefore, the methods used to describe the spherical model of the earth must be studied. There are two frequently used spherical models of the earth: one is a simple regular spherical model and the other is an ellipsoid model of the earth $[4,6,10]$.

### 2.4.1 Regular Spherical Model for Geolocation

The regular spherical model of the earth is a model taking the zero-altitude surface of the earth as a sphere with a given radius. Obviously, as the earth approximates an ellipsoid bulging at the equator and flattened at the poles, a regular spherical model will lead to a large model error.
One point to make is that in a regular spherical model of the earth, the transformation relation from its geodetic Cartesian coordinates to longitude, latitude, and altitude geodetic coordinates is

$$
\left.\begin{array}{l}
x=(R+H) \cos B \cos L  \tag{2.28}\\
y=(R+H) \cos B \sin L \\
z=(R+H) \sin B
\end{array}\right\}
$$

where $R$ is the radius of the zero-altitude surface of the earth and $(L, B, H)$ represents the longitude, latitude, and altitude of the emitter, respectively. The transformation from longitude, latitude, and altitude geodetic coordinates to geodetic Cartesian coordinates is

$$
\left\{\begin{array}{l}
L=\tan ^{-1}(y / x)  \tag{2.29}\\
B=\tan ^{-1}\left(z / \sqrt{x^{2}+y^{2}}\right) \\
H=\sqrt{x^{2}+y^{2}+z^{2}}-R
\end{array}\right.
$$

### 2.4.2 Ellipsoid Model of the Earth

An ellipsoid model of the earth takes the zero-altitude surface of the earth as a regular rotational ellipsoid. In fact, due to the uneven geological structure of the earth, the zero-altitude earth surface established based on the gravity potential is an irregular ellipsoid, so an errorless standard ellipsoid model is impossible. In order to describe the earth's surface more accurately and give an accurate description of longitude, latitude, and altitude of geodetic coordinates at one point on the earth's surface on that basis, it is necessary to adopt an earth ellipsoid as close to the
earth's surface as possible. At present, the description of the earth ellipsoid includes reference ellipsoids established in a reference-ellipsoid-centric geodetic coordinate system, such as those defined in the China Geodetic Coordinate System 1980 and in the Beijing Geodetic Coordinate System 1954, which give relatively accurate descriptions of regions and areas in China. The description also includes the earth ellipsoids established in geocentric geodetic coordinate systems, such as those defined in the WGS-72 and WGS-84 coordinates of the United States Department of Defense. All applications represented by GPS now use the WGS-84 coordinate system [9]. Therefore, the earth surface model used in this book is the WGS-84 earth ellipsoid surface in the WGS-84 coordinate system.

### 2.4.2.1 Definition of WGS-84 Earth Ellipsoid

The WGS-84 coordinate system is a Conventional Terrestrial System (CTS). The origin and axes of the WGS-84 coordinate system are defined as follows: the origin is the earth's center of mass; the $z$ axis points in the direction of the Conventional Terrestrial Pole (CTP), as defined by $\mathrm{BIH}_{1984.0}$; the $x$ axis points to the intersection of the plane of the zero meridian defined by $\mathrm{BIH}_{1984.0}$ and the plane of the CTP's equator; and the $y$ axis completes a right-handed coordinate system [4] with the $z$ axis and the $x$ axis.
The geometric center of the WGS-84 ellipsoid coincides with the earth's center of mass. The ellipsoid rotation axis coincides with the $z$ axis. The primary geometric parameters include:

$$
\begin{aligned}
& \text { Semi-major axis: } \quad a=6378137 \mathrm{~m} \pm 2 \mathrm{~m} \\
& \text { Square of first eccentricity: } \quad e^{2}=0.00669437999013
\end{aligned}
$$

The WGS-84 earth ellipsoid always relates to the WGS-84 coordinate system and the WGS-84 is in ECEF coordinates. Therefore, when we use the WGS-84 earth ellipsoid model, it means the WGS-84 coordinate system without any other explanation.

### 2.4.2.2 Transformation between WGS-84 Geodetic Coordinates and Cartesian Coordinates

The WGS-84 coordinate system is also a type of geodetic coordinate system. Its coordinates include longitude, latitude, and altitude geodetic coordinates and Cartesian coordinates. We often use longitude, latitude, and altitude geodetic coordinates to indicate the location of a point in the WGS-84 coordinate system, but the calculation of Cartesian coordinates is more convenient. Transformation between both coordinates is often used in geolocation [4].

1. Transformation from the WGS-84 geodetic coordinate system to the ECEF coordinate system

Transformation from longitude, latitude, and altitude geodetic coordinates to ECEF coordinates is as follows:

$$
\left.\begin{array}{l}
x=(N+H) \cos B \cos L  \tag{2.30}\\
y=(N+H) \cos B \sin L \\
z=\left[N\left(1-e^{2}\right)+H\right] \sin B
\end{array}\right\}
$$

where $N$ is the curvature radius of local prime vertical circle:

$$
\begin{equation*}
N=\frac{a}{\sqrt{1-e^{2} \sin ^{2} B}} . \tag{2.31}
\end{equation*}
$$

2. Transformation from the ECEF coordinate system to the WGS-84 geodetic coordinates system

Since the ellipsoid model is not isotropic, the transformation from Cartesian coordinates to longitude, latitude, and altitude coordinates is complex. An iterative calculation is therefore necessary.

From the first and the second expressions in Equation (2.30) we can obtain:

$$
\left.\begin{array}{l}
\tan L=\frac{y}{x}  \tag{2.32}\\
\sin L=\frac{y}{(N+H) \cos B}=\frac{y}{\sqrt{x^{2}+y^{2}}} \\
\cos L=\frac{x}{(N+H) \cos B}=\frac{x}{\sqrt{x^{2}+y^{2}}}
\end{array}\right\} .
$$

From the first and the third expressions in Equation (2.30) we can obtain:

$$
\begin{equation*}
\tan B=\frac{(N+H) \cos L z}{\left[N\left(1-e^{2}\right)+H\right] x} . \tag{2.33}
\end{equation*}
$$

By combining the third expression in Equation (2.32), we can rewrite the above formula as

$$
\begin{equation*}
B=\tan ^{-1}\left[\frac{z}{\sqrt{x^{2}+y^{2}}}\left(1-\frac{e^{2} N}{(N+H)}\right)^{-1}\right] \tag{2.34}
\end{equation*}
$$

From the third expression in Equation (2.32),

$$
\begin{equation*}
H=\frac{\sqrt{x^{2}+y^{2}}}{\cos B}-N . \tag{2.35}
\end{equation*}
$$

Evaluate Equations (2.34) and (2.35) using the iterative method. Before starting the iteration procedure, set

$$
\left.\begin{array}{l}
N_{0}=a  \tag{2.36}\\
H_{0}=\sqrt{x^{2}+y^{2}+z^{2}}-\sqrt{a b} \\
B_{0}=\tan ^{-1}\left[\frac{z}{\sqrt{x^{2}+y^{2}}}\left(1-\frac{e^{2} N_{0}}{\left(N_{0}+H_{0}\right)}\right)^{-1}\right]
\end{array}\right\}
$$

where $b$ is the minor axis of the ellipsoid meridian plane. Then use the following formulas in each iteration:

$$
\left.\begin{array}{l}
N_{i}=\frac{a}{\sqrt{1-e^{2} \sin ^{2} B_{i-1}}}  \tag{2.37}\\
H_{i}=\frac{\sqrt{x^{2}+y^{2}}}{\cos B_{i-1}}-N_{i} \\
B_{i}=\tan ^{-1}\left[\frac{z}{\sqrt{x^{2}+y^{2}}}\left(1-\frac{e^{2} N_{i}}{\left(N_{i}+H_{i}\right)}\right)^{-1}\right]
\end{array}\right\}
$$

until

$$
\left|H_{i}-H_{i-1}\right|<\varepsilon_{1}
$$

and

$$
\left|B_{i}-B_{i-1}\right|<\varepsilon_{2},
$$

where $\varepsilon_{1}$ and $\varepsilon_{2}$ are determined by the required accuracy.

### 2.5 Coverage Area of a Satellite

The ground coverage of an artificial satellite is also called the visible ground area of the satellite [1]. In carrying out an electronic reconnaissance task, the satellite will first judge whether the emitter is within the coverage area of the satellite.

### 2.5.1 Approximate Calculation Method for the Coverage Area

Only ground coverage of the satellite at a point in the orbit is considered, without taking into account of the satellite antenna beam direction and size and specific pointing. In actual projects, the coverage calculation is complex. The coverage can be calculated and analyzed through the simulation software Satellite Tool $\mathrm{Kit}^{\circledR}{ }^{\circledR}(\mathrm{STK})$. In order to simplify the calculation process of satellite coverage, a regular spherical model of the earth can be approximately used to make an approximate calculation. Suppose that the earth is a round sphere with a radius equal to $R_{e}$ and $\mathrm{O}_{\mathrm{e}}$ is the center of the earth. The height of the satellite at any point in the orbit from the ground is $h$ and the subsatellite point is $S$, as shown in Figure 2.8. Also suppose that the electromagnetic waves propagate in straight lines and the refraction of atmosphere is not considered. In the figure, draw two tangent lines between satellite O and the spherical surface, with points of tangency as $\mathrm{P}_{1}$ and $\mathrm{P}_{2}$. The geocentric angle $d$ is called the coverage angle. By taking $90^{\circ}-d$ as a half cone angle, $\mathrm{OO}_{\mathrm{e}}$ as the axis, and $\mathrm{OP}_{2}$ as the generatrix, draw a normal cone tangent with the earth. The ground area above the tangent lines is the coverage area.
It can be seen from the right triangle $\mathrm{OO}_{\mathrm{e}} \mathrm{P}_{1}$ that the coverage angle $d$ is

$$
\cos d=\frac{R_{e}}{R_{e}+h}
$$

The angle $\psi$ covered by the satellite antenna beam is

$$
\sin \frac{\psi}{2}=\frac{R_{e}}{R_{e}+h}
$$

The corresponding coverage area $A_{s}$ is

$$
\begin{equation*}
A_{s}=\pi R_{e}^{2}\left(\frac{2 h}{R_{e}+h}\right) . \tag{2.38}
\end{equation*}
$$

The above-mentioned calculation takes the tangent of the earth's surface as the reference. In fact, this method is not allowed in project applications. If the elevation angle of the ground emitter is too low, lots of ground reflection and blocking will seriously affect the receiving of


Figure 2.8 Diagram of the satellite coverage area
transmitting signals. In order to achieve a result in receiving of the reconnaissance satellite, generally the angle of sight between the satellite and ground emitter should be more than a given angle $\sigma$, which is the minimum observation angle, as shown in Figure 2.8. With the restriction of a minimum observation angle, the satellite coverage area will decrease.
According to the relationship as shown in the figure, by making derivation by applying the sine theorem, we can obtain the coverage angle $d^{\prime}$ under the restriction of a minimum observation angle:

$$
\begin{equation*}
d^{\prime}=\arccos \frac{R_{e} \cos \sigma}{R_{e}+h}-\sigma . \tag{2.39}
\end{equation*}
$$

The coverage area can be obtained as

$$
\begin{equation*}
A_{S}=\pi R_{e}^{2}\left(2-2 \cos d^{\prime}\right) \tag{2.40}
\end{equation*}
$$

It can be seen from the above expression that the satellite coverage area is larger as its distance from the ground is greater.

### 2.5.2 Examples of Calculation of the Coverage Area

Suppose the minimum observation angle $\sigma=30^{\circ}$; according to expression (2.40) the relationship between the radius of the instantaneous coverage area of the satellite and satellite altitude can be obtained, as shown in Figure 2.9. It can be seen from this that a satellite with an altitude of about $500-1000 \mathrm{~km}$ can have a coverage radius of about $2500-3000 \mathrm{~km}$.
The above-mentioned discussion is about ground coverage of a single satellite in the stationary state or instantaneous state. In fact, the satellite generally moves in an orbit, and the coverage area on the ground will sweep over a strip of coverage with the movement of the satellite. For example, for a sunsynchronous orbit with an orbital altitude of 1000 km , by supposing the view-angle coverage of the sensor on the satellite is $90^{\circ}$, we can obtain a coverage area in an instantaneous state, as shown by the dotted area in Figure 2.10.


Figure 2.9 Relationship between the satellite coverage radius and satellite altitude


Figure 2.10 Single-satellite coverage area $(H=100 \mathrm{~km})$ calculated in $\mathrm{STK}^{\circledR}$

In order to meet continuous regional earth coverage requirements, a single satellite is not enough. A multisatellite constellation is used to meet continuous coverage requirements for hotspot area reconnaissance. Then the subsatellite track of multiple satellites on the earth's surface constitutes a subsatellite 'belt' zone.


Figure 2.11 Reconnaissance diagram of a satellite side-looking antenna

### 2.5.3 Side Reconnaissance Coverage Area

In the above-mentioned calculation of the satellite coverage area, we supposed that the satellite used omnidirectional antenna and had no pattern constraint in earth coverage. However, in actual coverage, the antenna pattern will be taken into consideration. In some cases, pointing the antenna of the ground emitter is not in a direction perpendicular to the ground (which is different from satellite communication and tracking telemetry and command antennas), but has a small angle of elevation with the horizontal plane of the ground. If pointing downwards to the earth's center, the main beams of electronic reconnaissance satellite antennas may point to the side lobes of the emitter signal (such as the radar) or the main beams of the emitter may point to the side lobes of reconnaissance antennas. In order to increase the reconnaissance distance, one option is to install reconnaissance antennas in an offset position, as shown in Figure 2.11.
Suppose that the elevation angle of deviation is $45^{\circ}$. We can obtain the coverage area of the satellite by STK, as shown by the dotted area in Figure 2.12. At this moment, satellite coverage is not bilaterally symmetric. A crescent gap appears at one side. With an offset installation, the satellite's reconnaissance range is $1.4-5$ times larger than that in the original vertically down installation of the antenna. The signal has an attenuation of $3-14 \mathrm{~dB}$ in free space, but the alignment of the main lobes of satellite emitter antennas brings about 20-30 dB gains. Therefore, it may still obtain a better electronic reconnaissance and interception SNR (signal-to-noise ratio).

### 2.6 Fundamentals of Geolocation

As viewed from geometry, a point in space can be determined by intersection of three or more surfaces or planes in 3D space. Geolocation parameters or measurements received by an electronic reconnaissance receiver from a target emitter, for example, azimuth angle $\beta$ or $\varphi$, elevation angle $\varepsilon$, direction cosine $l, m, n$, slant range $r$, range sum $\rho$ or $s$, range difference $\Delta r$, and altitude $h$, correspond to a plane or surface in geometry, respectively. The plane corresponding to geolocation parameters of the same emitter obtained through the detection system is defined as a geolocation plane. Through some combinations, two planes intersect in a line


Figure 2.12 Coverage area of side reconnaissance
or two lines, or a plane and line intersect at a point. The point of the emitter location can be determined according to the line and the point obtained therefrom. The line of intersection of planes is the line of position (LOP), and the point of intersection of lines or a line and plane is a geolocation point $[9,11]$.

### 2.6.1 Spatial Geolocation Plane

Suppose the location $\mathbf{X}$ of the target emitter is $\mathbf{X}=\left[\begin{array}{lll}x & y & z\end{array}\right]^{\mathrm{T}}$ and the station location $\mathbf{X}_{i}$ of the reconnaissance receiver is $\mathbf{X}_{i}=\left[x_{i}, y_{i}, z_{i}\right]^{\mathrm{T}}$. The spatial geolocation plane or surface corresponding to the observations obtained by the electronic reconnaissance receiver, such as the azimuth angle, elevation angle, direction cosine, range sum and range difference, and their algebraic expressions, are shown in Table 2.1.

### 2.6.2 Spatial Line of Position (LOP)

If one receiver can measure two observations at the same time, for example, azimuth and elevation ( $\phi$ and $\varepsilon$ ) or azimuth and slant range ( $\phi$ and $r$ ), and if two receivers can measure azimuth angle $\left(\varphi_{1}, \varphi_{2}\right)$ of one emitter, the geolocation planes corresponding to two measurements will intersect in a space curve, and the emitter will be located in this line. Therefore the line is called the LOP. Examples are shown in Table 2.2.
According to the above geometric analysis, at least three or more geolocation planes should be provided to realize the 3D space geolocation of the emitter. Geometric fundamentals of space geolocation as discussed above play an important and useful role in understanding geolocation theory, realizability, and the law that geolocation error caused by measurement error varies with the spatial location of the emitter.

Table 2.1 List of geolocation planes

| Observations | Form of geolocation plane | Algebraic expression |
| :--- | :--- | :--- |
| Azimuth angle $\beta, \phi$ | $\operatorname{tg} \beta=\frac{x-x_{i}}{y-y_{i}}$ |  |
| $\operatorname{tg} \phi=\frac{y-y_{i}}{x-x_{i}}$ |  |  |
| or |  |  |
| $\cos \beta\left(x-x_{i}\right)-\sin \beta\left(y-y_{i}\right)=0$ |  |  |
| $\sin \phi\left(x-x_{i}\right)-\cos \phi\left(y-y_{i}\right)=0$ |  |  |

Plane

Elevation angle $\varepsilon$


$$
\begin{aligned}
\operatorname{tg\varepsilon } & =\frac{z-z_{i}}{\sqrt{\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}}} \\
& =\frac{z-z_{i}}{d}
\end{aligned}
$$

where $d$ is horizontal range of emitter, or $\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}$ $-\operatorname{ctg}^{2} \varepsilon\left(z-z_{i}\right)^{2}=0$

Direction cosine $l, m, n$

$l=\cos \alpha=\frac{x-x_{i}}{r}$
$m=\cos \beta=\frac{y-y_{i}}{r}$
$n=\cos \gamma=\frac{z-z_{i}}{r}=\sqrt{1-l^{2}-m^{2}}$
where
$r=\sqrt{\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}+\left(z-z_{i}\right)^{2}}$

Cosines in three directions correspond to three circular conical surfaces with $x, y$, and $z$ axes as conical axes and $\alpha, \beta$, and $\gamma$ as semi-vertical angles, respectively. Direction line $\gamma$ from intersection of two planes is the LOP

Table 2.1 (continued)

| Observations | Form of geolocation plane | Algebraic expression |
| :---: | :---: | :---: |
| Slant range $r$ | ${ }^{\text {z }}$ | $\begin{aligned} r= & {\left[\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}\right.} \\ & \left.+\left(z-z_{i}\right)^{2}\right]^{1 / 2} \end{aligned}$ |
|  |  | or |
|  |  | $\begin{aligned} r= & l\left(x-x_{i}\right) \\ & +m\left(y-y_{i}\right)+n\left(z-z_{i}\right) \end{aligned}$ |
|  |  | where |
|  |  | $\begin{aligned} & l=\cos \phi \cos \varepsilon=\sin \beta \cos \varepsilon \\ & m=\sin \phi \cos \varepsilon=\cos \beta \cos \varepsilon \\ & n=\sin \varepsilon \end{aligned}$ |

Spherical surface
Range sum $\rho$ or $s$

Rotary hyperboloid

| Altitude $h$ | An ellipsoid with equal <br> altitude $h$ above the <br> earth surface, or a plane <br> that can be seen parallel <br> to the $x y$ plane in a small <br> area | WGS-84 ellipsoid or $h=z-z_{i}$ |
| :--- | :--- | :--- |

Table 2.2 Examples of LOP

| Measurements | Form of LOP |
| :--- | :--- | | Description |
| :--- |
| Azimuth $\beta, \phi ;$ <br> elevation $\varepsilon$ |
| A LOP obtained from intersection <br> of azimuth plane and elevation <br> cone points in the emitter <br> direction, or is called direction <br> vector. Direction $\operatorname{cosines~of~this~}$ <br> vector are: <br> $l=\cos \phi \cos \varepsilon=\sin \beta \cos \varepsilon$ <br> $m=\sin \phi \cos \varepsilon=\cos \beta \cos \varepsilon$ <br> $n=\sin \varepsilon$ |

Direction vector

Azimuth $\phi$, slant range $r$


LOP of direction slant range
Azimuth $\phi_{i}$, azimuth $\phi_{j}$

| Slant range $r_{i}$, <br> range sum <br> $\rho=r_{i}+r_{j}$ |
| :--- |
| When $r_{i}=\Delta r$ or $d+\Delta r$, <br> $r_{i}=\Delta r+d$ or $\Delta r$, and <br> $\rho=r_{i}+r_{j}=d+2 \Delta r$, the slant <br> range spherical surface and range <br> sum ellipsoid intersect at a point a <br> or a ${ }^{\prime} ;$ when $\Delta r<r_{i}<d+2 \Delta r$ and <br> $\rho=r_{i}+r_{j}=d+2 \Delta r$ (fixed value), <br> a circle will be obtained from the <br> intersection of the spherical <br> surface and ellipsoid |

A straight line parallel to the $z$ axis is obtained from intersection of azimuth plane $\phi_{i}$ of station $X_{i}$ and azimuth plane $\phi_{j}$ of station $X_{j}$. Its expression is:
Any $z_{T}$ value $\left\{\begin{array}{l}x=x_{T} \\ y=y_{T}\end{array}\right.$

After intersection of azimuth plane and slang range spherical surface, a spatial semicircle arc in azimuth plane, with start and end points at $\pm r$ in $z$ axis is obtained. If $X_{i}$ is located at the ground, the LOP is a circular arc of $90^{\circ}$


Geolocating straight line

$r_{i}, \rho$ LOP

When $r_{i}=\Delta r$ or $d+\Delta r$,
$r_{i}=\Delta r+d$ or $\Delta r$, and
$\rho=r_{i}+r_{j}=d+2 \Delta r$, the slant range spherical surface and range sum ellipsoid intersect at a point a or $\mathrm{a}^{\prime}$; when $\Delta r<r_{i}<d+2 \Delta r$ and $\rho=r_{i}+r_{j}=d+2 \Delta r$ (fixed value), a circle will be obtained from the intersection of the spherical surface and ellipsoid

### 2.7 Measurement Index of Geolocation Errors

In practice, measurement of geolocation parameters always contain some errors that are affected by various factors. A practical location estimated by the geolocation system is always prone to error. The size and distribution of geolocation errors are closely correlated to the geometric scenario, observability, geolocation methods, and parameter measurement error, which is one of the important technical specifications of the geolocation system. Hence, the research on geolocation error plays an important role.

### 2.7.1 General Definition of Error

Suppose the true location of the emitter is $\mathbf{x}=\left[\begin{array}{ll}x & y \\ z\end{array}\right]^{\mathrm{T}}$, there are $i$ observations $\mathbf{z}_{1}, \ldots, \mathbf{z}_{i}$ obtained through measurement, and the estimated result of a geolocation is $\widehat{\mathbf{x}}$, which is generally the function of observations, that is, $\widehat{\mathbf{x}}=\mathbf{f}\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{i}\right)$. Then the geolocation error is expressed as

$$
\begin{equation*}
\widetilde{\mathbf{x}}=\widehat{\mathbf{x}}-\mathbf{x} \tag{2.41}
\end{equation*}
$$

The measurement error is usually random, and so is the geolocation error. Hence, each geolocation estimation is different.

### 2.7.1.1 Bias of Geolocation Error

The bias of estimation is expressed as [12]

$$
\begin{equation*}
\mathbf{x}_{\text {bias }}=E[\hat{\mathbf{x}}]-\mathbf{x}, \tag{2.42}
\end{equation*}
$$

where $E[\cdot]$ means the expectation of a random vector. If multiple repeated geolocation estimates under the same conditions are made, the estimation bias can be approximated as

$$
\begin{equation*}
\mathbf{x}_{\text {bias }}=\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N}\left(\widehat{\mathbf{x}}_{n}-\mathbf{x}\right) \tag{2.43}
\end{equation*}
$$

where $\widehat{\mathbf{x}}_{n}$ is the result of the nth geolocation. Accurate bias can be obtained when $N$ approaches infinity. In fact, people always expect that there is no bias in a geolocation result, that is, $E[\widehat{\mathbf{x}}]=$ $\mathbf{x}$. If there is bias in the estimated result, it may be system bias in measurement parameters or estimation bias in the estimation algorithm. However, some geolocation algorithms is not unbiased. We may be content with the suboptimal, that is, the algorithm is asymptotically unbiased. In another words, when the times of measurement meet infinity, the error estimation is unbiased:

$$
\begin{equation*}
E\left[\lim _{i \rightarrow \infty} \widehat{\mathbf{x}}\left(\mathbf{z}_{1}, \ldots, \mathbf{z}_{i}\right)\right]=\mathbf{x} \tag{2.44}
\end{equation*}
$$

### 2.7.1.2 Variance and Root Mean Square (RMS) of Geolocation Error

Another important index is the variance and root mean square error. The location is generally a vector, so the location error is not a scalar, but a vector. It is more suitable to use the covariance matrix to describe the error. The covariance matrix is defined as [12]

$$
\begin{equation*}
\mathbf{P}=E\left[(\hat{\mathbf{x}}-\mathbf{x})(\hat{\mathbf{x}}-\mathbf{x})^{\mathrm{T}}\right] . \tag{2.45}
\end{equation*}
$$

Sometimes we still pay more attention to the range error of location, which is usually called the range variance, expressed as

$$
\begin{equation*}
\sigma^{2}=E\left[(\hat{\mathbf{x}}-\mathbf{x})^{\mathrm{T}}(\hat{\mathbf{x}}-\mathbf{x})\right]=\operatorname{tr}(\mathbf{P}) \tag{2.46}
\end{equation*}
$$

where $\operatorname{tr}(\cdot)$ indicates the operation of solving the matrix trace, which is equal to the sum of diagonal elements of a matrix. The corresponding RMSE (root mean square error) can be expressed as [11]

$$
\begin{equation*}
\sigma=\sqrt{\operatorname{tr}(\mathbf{P})} \tag{2.47}
\end{equation*}
$$

It is also called the standard deviation if the estimation is unbiased.
After multiple repeated geolocation estimation tests, the empirical estimation of the geolocation error covariance can be obtained:

$$
\begin{equation*}
\widehat{\mathbf{P}}_{N}=\frac{1}{N} \sum_{n=1}^{N}\left(\widehat{\mathbf{x}}_{n}-\mathbf{x}\right)\left(\widehat{\mathbf{x}}_{n}-\mathbf{x}\right)^{\mathrm{T}} . \tag{2.48}
\end{equation*}
$$

Then, the range RMSE is [12]

$$
\begin{equation*}
\hat{\sigma}_{N}=\sqrt{\frac{1}{N} \sum_{n=1}^{N}\left(\hat{\mathbf{x}}_{n}-\mathbf{x}\right)^{\mathrm{T}}\left(\hat{\mathbf{x}}_{n}-\mathbf{x}\right)} \tag{2.49}
\end{equation*}
$$

If $N$ is large enough, obviously we can obtain $\mathbf{P}=\lim _{N \rightarrow \infty} \widehat{\mathbf{P}}_{N}$ and $\sigma=\lim _{N \rightarrow \infty} \widehat{\sigma}_{N}$. Note that

$$
\hat{\sigma}_{N} \neq \frac{1}{N} \sum_{n=1}^{N} \sqrt{\left(\widehat{\mathbf{x}}_{n}-\mathbf{x}\right)^{\mathrm{T}}\left(\widehat{\mathbf{x}}_{n}-\mathbf{x}\right)}
$$

### 2.7.1.3 Relative Range Error

Generally speaking, the geolocation error of the geolocation system increases as the range from the receiver increases. For example, for an active radar with a higher ranging accuracy, when the range is long the geolocation error is $\sigma \approx r \sigma_{\theta}$, which shows that the range is basically directly proportional to the geolocation error, that is, the relative range error parameter is independent of the range. The relative range error is an important index for measuring accuracy of an active radar. Similarly, the passive geolocation system can use the concept of the relative range error from active radar system specifications for reference. It is defined as [11]

$$
\text { Relative range error } \quad(\% \mathrm{R})=\frac{\sigma}{r} \times 100 \%
$$

For a single reconnaissance station, the reference point for calculating range $r$ is the reconnaissance station itself; for multiple observation stations, the reference point for calculating $r$ can be the geometric center of multiple reconnaissance stations or one of these stations.
As for error in reconnaissance geolocation, the geolocation error may not be directly proportional to the emitter range. Therefore the relative geolocation error is generally a nonlinear function of the emitter range and azimuth.

### 2.7.2 Geometrical Dilution of Precision (GDOP)

Due to different locations of the target emitters, the uncertainty area for different emitter positions with the same error of geolocation parameter measurement error may be different. Geolocation error is also a function of the emitter location. In order to describe this relationship better, the term geometrical dilution of precision (GDOP) is used in engineering, which is expressed as [9]

$$
\begin{gather*}
G D O P(x, y)=\sqrt{\sigma_{x}^{2}+\sigma_{y}^{2}}, \quad \text { for 2D case }  \tag{2.50}\\
G D O P(x, y, z)=\sqrt{\sigma_{x}^{2}+\sigma_{y}^{2}+\sigma_{z}^{2}}, \quad \text { for 3D case. } \tag{2.51}
\end{gather*}
$$

GDOP is a factor that describes the distribution of geolocation errors. It can be expressed either in RMS (root mean square) in expressions (2.50) and (2.51) or in circular error probability (CEP) in Section 2.7.3.
In order to give a more direct viewing representation of the emitter GDOP, GDOP in one area can be plotted in the form of a contour map on which contour values are indicated as the geolocation error.

### 2.7.3 Graphical Representation of the Geolocation Error

Factors giving rise to the geolocation error may be diversified. According to the central limit theorem [13], as the common effect of multiple independent factors affects the geolocation error, the geolocation error usually approaches an approximate normal distribution. A statistical property of the distribution is expressed by first and secondary moments of the distribution function. We can use the joint Gaussian distribution to describe approximately the distribution of the geolocation error in 3D space.
If in 3D space the location error follows the normal distribution, the following expression can be used to express its spatial probability density distribution [12], that is,

$$
\begin{equation*}
p(\mathbf{x})=\frac{1}{(\sqrt{2 \pi})^{3}|\mathbf{P}|^{1 / 2}} \exp \left\{-\frac{1}{2}(\mathbf{x}-\overline{\mathbf{x}})^{\mathrm{T}} \mathbf{P}^{-1}(\mathbf{x}-\overline{\mathbf{x}})\right\} \tag{2.52}
\end{equation*}
$$

where the error vector is $\mathbf{x}=\left[\begin{array}{ll}x & y \\ \hline\end{array}\right]^{\mathrm{T}}$, the mean error vector is $\overline{\mathbf{x}}=E[\mathbf{x}]$, and the covariance matrix of error vector $\mathbf{P}$ is a symmetric and positive matrix, that is,

$$
\mathbf{P}=E\left[(\mathbf{x}-\overline{\mathbf{x}})(\mathbf{x}-\overline{\mathbf{x}})^{\mathrm{T}}\right]=\left[\begin{array}{ccc}
\sigma_{x}^{2} & \rho_{x y} \sigma_{x} \sigma_{y} & \rho_{x y} \sigma_{x} \sigma_{z}  \tag{2.53}\\
\rho_{x y} \sigma_{x} \sigma_{y} & \sigma_{y}^{2} & \rho_{y z} \sigma_{y} \sigma_{z} \\
\rho_{x y} \sigma_{x} \sigma_{z} & \rho_{y z} \sigma_{y} \sigma_{z} & \sigma_{z}^{2}
\end{array}\right] .
$$

As for 2D geolocation, suppose that the geolocation error follows a 2D normal distribution. Its probability density function is

$$
\begin{equation*}
p(\mathbf{x})=\frac{1}{2 \pi|\mathbf{P}|^{1 / 2}} \exp \left\{-\frac{1}{2}(\widehat{\mathbf{x}}-\mathbf{x})^{\mathrm{T}} \mathbf{P}^{-1}(\widehat{\mathbf{x}}-\mathbf{x})\right\} . \tag{2.54}
\end{equation*}
$$

Sometimes it may be too complicated to describe the geolocation error with such a complex probability density function. In order to simplify the description, we may use a confidence ellipse related to the probability $p(\mathbf{x})$, which can also be called the elliptical error probability
(EEP). The size and shape of the ellipse indicates the geolocation error. The longer the major axis $a$ and minor axis $b$, larger the ellipse is and the worse the geolocation quality becomes (the larger the geolocation error is).
Since the covariance matrix $\mathbf{P}$ is sure to be a real symmetric positively definite matrix, according to matrix analysis, by decomposing eigenvalues of the covariance matrix $\mathbf{P}$ we can obtain

$$
\begin{equation*}
\mathbf{P}=\mathbf{U} \Sigma \mathbf{U}^{\mathrm{T}} \tag{2.55}
\end{equation*}
$$

where $\mathbf{U}$ is a unitary matrix, meeting the characteristic $\mathbf{U} \mathbf{U}^{\mathrm{T}}=\mathbf{I}$ and $\Sigma=\operatorname{diag}\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}\right\}$ is the diagonal matrix of its eigenvalues. We can consider the above eigenvalue decomposition process as a process of rotation of coordinates. The unitary matrix $\mathbf{U}$ is a coordinate rotation matrix and the obtained eigenvalues are three axes of the ellipse.
As for 2D geolocation, the eigenvalue of the location covariance matrix [9] can be calculated as

$$
\begin{equation*}
\lambda_{1,2}=\frac{\sigma_{x}^{2}+\sigma_{y}^{2} \pm \sqrt{\left(\sigma_{x}^{2}-\sigma_{y}^{2}\right)^{2}+4 \sigma_{x y}^{2}}}{2} . \tag{2.56}
\end{equation*}
$$

The semimajor axis of a $1 \sigma$ error ellipse (defined as the error ellipse when the amplification factor $k=1)$ is $\max \left(\sqrt{\lambda_{1}}, \sqrt{\lambda_{2}}\right)$, the semiminor axis is $\min \left(\sqrt{\lambda_{1}}, \sqrt{\lambda_{2}}\right)$, and the inclination of the semi-major axis of the ellipse relative to the $x$ axis is

$$
\begin{equation*}
\theta=\frac{1}{2} \operatorname{arctg} \frac{2 \rho \sigma_{x} \sigma_{y}}{\sigma_{x}^{2}-\sigma_{y}^{2}} . \tag{2.57}
\end{equation*}
$$

The size of the location uncertainty can also be measured by the area of the error ellipse $\pi \lambda_{1} \lambda_{2}$. An EEP example is shown in Figure 2.13, where points of the geolocation error are distributed on the earth's plane according to a variance matrix. By decomposing the eigenvalue of the error matrix we can obtain a $1 \sigma$ error ellipse as shown in the figure.
Reference [12] gives another expression of the semi-major axis and semi-minor axis of the ellipse under the confidence level:

$$
\begin{align*}
& a^{2}=2 \frac{\sigma_{x}^{2} \sigma_{y}^{2}-\rho_{x y}^{2} \rho_{x}^{2} \rho_{y}^{2}}{\sigma_{x}^{2}+\sigma_{y}^{2}-\sqrt{\sigma_{x}^{2}-\sigma_{y}^{2}+4 \rho_{x y}^{2} \rho_{x}^{2} \rho_{y}^{2}}} C^{2},  \tag{2.58}\\
& b^{2}=2 \frac{\sigma_{x}^{2} \sigma_{y}^{2}-\rho_{x y}^{2} \rho_{x}^{2} \rho_{y}^{2}}{\sigma_{x}^{2}+\sigma_{y}^{2}+\sqrt{\sigma_{x}^{2}-\sigma_{y}^{2}+4 \rho_{x y}^{2} \rho_{x}^{2} \rho_{y}^{2}}} C^{2} \tag{2.59}
\end{align*}
$$

where $C=-2 \ln \left(1-P_{e}\right)$ and $P_{e}$ is the confidence of the emitter in the error ellipse (e.g., 0.5 represents $50 \%$ and 0.9 represents $90 \%$ ).

### 2.7.4 Spherical Error Probability (SEP) and Circular Error Probability (CEP)

In actual applications, it may still be inconvenient to use more parameters such as the major axis, minor axis, and direction to describe such an oblique ellipse. In an error analysis of


Figure 2.13 EEP geolocation error distribution


Figure 2.14 Relationship between CEP and EEP
passive geolocation, it is most common to use CEP to describe geolocation error, as shown in Figure 2.14 [9, 14, 15].
CEP means the radius of a circle, centered in the mean value of geolocation estimation points, within which half of the geolocation estimation points are expected to fall. CEP is


Figure 2.15 Relationship between SEP and CEP and $\sigma_{y} / \sigma_{x}$ and $\sigma_{z} / \sigma_{x}$
defined as $[9,16]$

$$
\begin{equation*}
\int_{0}^{C E P} p(r) \mathrm{d} r=0.5 \tag{2.60}
\end{equation*}
$$

The original concept of CEP evolved from artillery firing. That is to say, in the case where the geolocation is repeated 100 times, half of them will fall within the CEP circle and the other will fall out of the circle. In other words, if a point is located, the true emitter must have a $50 \%$ probability of falling within the circle with this point as the center and CEP as the radius.
As for 3D space, the description of error is not a circle, but an error sphere, which is called the spherical error probability (SEP).
Johnson et al. calculated the relationship between SEP and CEP and $\sigma_{y} / \sigma_{x}$ and $\sigma_{z} / \sigma_{x}$ under different 3D ellipsoid axis ratios in the numerical integration method, as shown in Figure 2.15 [ 9,16$]$. During the calculation, an assumption of $\sigma_{x} \geq \sigma_{y} \geq \sigma_{z}$ was used.
As shown in Figure 2.15, the actual shape of the error distribution is an ellipse. CEP, in fact, is an approximate expression. From the figure it can be seen that as for Gaussian distribution, here the error is no more than $10 \%$. CEP can be approximately expressed as $[9,12,16]$

$$
\begin{equation*}
C E P \approx 0.75 \sqrt{\sigma_{x}^{2}+\sigma_{y}^{2}} \tag{2.61}
\end{equation*}
$$

The radius of CEP can describe the size of the geolocation error, but the error distribution could not be indicated.

### 2.8 Observability Analysis of Geolocation

Observability is a measure for how well the unique initial state of a known system can be determined by using known finite inputs and outputs. In geolocation, this means determining whether the system has a unique solution. In geolocation practice, it shows whether the unique solution of the emitter location can be obtained based on observations. In space electronic reconnaissance geolocation, the observable condition under which a unique solution of emitter location can be obtained is one of the key issues researchers must make clear.
To assess whether the system is observable, we can suppose that each observation is free of noise, because essentially the noise does not affect observability of geolocation. In this case the observation equation is nonlinear, according to the observability theorem developed by Sun et al. [9].

Theorem 2.1 With respect to a nonlinear system with:

$$
\begin{align*}
& \text { State definition: } \quad \dot{\mathbf{x}}(t)=f(\mathbf{x}(t), t) ; \mathbf{x}\left(t_{0}\right)=\mathbf{x}_{0},  \tag{2.62}\\
& \text { Measurement definition }: \quad \mathbf{y}(t)=h(\mathbf{x}(t), t) . \tag{2.63}
\end{align*}
$$

If for all $x_{0}$ in convex set $\mathbf{S} \in \mathbf{R}^{n}$,

$$
\begin{equation*}
\mathbf{M}\left(x_{0}\right)=\int_{t_{0}}^{t_{1}} \Phi^{\mathrm{T}}\left(\tau, t_{0}\right) \mathbf{H}^{\mathrm{T}}(\tau) \mathbf{H}(\tau) \Phi\left(\tau, t_{0}\right) \mathrm{d} \tau \tag{2.64}
\end{equation*}
$$

which is positive definite, the system is completely observable in $S$, where

$$
\mathbf{H}(t)=\frac{\partial \mathbf{h}(\mathbf{x}, t)}{\partial \mathbf{x}}
$$

and $\boldsymbol{\Phi}\left(t, t_{0}\right)$ are transition matrix of $\partial \mathbf{f} / \partial \mathbf{x}$.
The above-mentioned theorem is for a continuous system. If sampling discretization of a continuous system, consider using the discrete observation sequence $\mathbf{z}_{i+n-1}=$ $\left\{z_{i}, z_{i+1}, \ldots, z_{i+n-1}\right\}$ to determine the system state $\mathbf{x}_{i}$ at time $i$. The conclusion equivalent to the above nonlinear observability theorem is as follows [9].

Theorem 2.2 For an $n$-dimension vector $\mathbf{x}_{k 0}^{*}$ in the initial set $\mathbf{S}$, given

$$
\boldsymbol{\Gamma}(i, i+N-1)=\left[\begin{array}{c}
\mathbf{H}_{i} \\
\mathbf{H}_{i+1} \boldsymbol{\Phi} \\
\ldots \\
\mathbf{H}_{i+n-1} \boldsymbol{\Phi}^{n-1}
\end{array}\right],
$$

where

$$
\mathbf{H}_{j}=\left.\frac{\partial \mathbf{h}_{j}\left(\mathbf{x}_{j}\right)}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_{j}}
$$

is the Jacobian matrix. If there is a positive integral $N$ making the rank of $\boldsymbol{\Gamma}(i, i+N-1)$, then

$$
\begin{equation*}
\operatorname{rank} \boldsymbol{\Gamma}(i, i+N-1)=n \tag{2.65}
\end{equation*}
$$

The system is completely observable in $\mathbf{S}$.

For example, according to the studies of Becker [17], necessary and sufficient conditions of observability for a bearings-only geolocation are

$$
\mathbf{x}(t)=\left[\begin{array}{l}
x(t)  \tag{2.66}\\
y(t)
\end{array}\right] \not \equiv \alpha(t) \mathbf{B} \mathbf{t}
$$

where $\alpha(t)$ is an arbitrary normalized function, $\mathbf{t}=\left(1\left(t-t_{0}\right) \cdots\left(t-t_{0}\right)^{N}\right)^{\mathrm{T}}$ and $\mathbf{B}$ is an arbitrary $2(N+1)$ matrix independent of $\mathbf{t}$.
Since the satellite is running in a constraint orbit in accordance with regular laws, an existing observable conclusion of an angle-measuring geolocation does not directly apply to the satellite geolocation. The issue on geolocation observability in satellite electronic reconnaissance is related to a specific application scenario and what kind of geolocation parameters are used.

## References

1. Wang, Y. and Liu, Y. (2003) Military Satellite and Application Concept. Beijing: National Defence Industry Press, May 2003 (in Chinese).
2. Dennis, R. Satellite Communication, 3rd edn. McGraw-Hill Inc.
3. Zhou, Z. and Li, Q. (1998) Modern Aerospace Measurement and Control Principles. Changsha: NUDT Publish House, September 1998 (in Chinese).
4. Wang, W. and Yu, Z. (2007) Orbit Determination of Spacecraft - Model and Algorithm. Beijing: National Defence Industry Press, January 2007 (in Chinese).
5. Ren, X. (1998) Orbit Dynamics of Sputnik. Changsha: NUDT Publish House (in Chinese).
6. Zhong, D. (2002) Research on Three-Satellite TDOA Geolocation Based on WGS-84. Changsha: Graduate School of National University of Defense Technology, November 2002 (in Chinese).
7. Liu, L., Wu, B., and Yang, P. (2005) Orbit Accuracy Determination and Self-Calibration Technique of Spacecraft. Beijing: National Defence Industry Press, January 2005 (in Chinese).
8. Zhang, R. (1998) The Kinetics and Control of Attitude for Satellite Orbit. Beijing: Beihang University Press, May 1998 (in Chinese).
9. Sun, Z., Zhou, Y., and He, L. (1996) Active and Passive Localization Technology with Single or Multiple Base. Beijing: National Defence Industry Press, May 1996 (in Chinese).
10. Xi, X., Wang, W., and Gao, Y. (2003) Fundamentals of Near-Earth Spacecraft Orbit. Changsha: NUDT Publish House, (in Chinese).
11. Sun, Z., Guo, F., Feng, D., et al. (2008) Passive Localization and Tracking Technology by Single Observer. Beijing: National Defence Industry Press, November 2008 (in Chinese).
12. Steven, K. (1998) Fundamentals of Statistical Signal Processing, Volume I: Estimation Theory/Volume II: Detection Theory, Prentice Hall PTR.
13. Bar-Shalom, Y., Li, X.R. and Kjrubarajan, T. (2001) Estimation with Applications to Tracking and Navigation, John Wiley \& Sons, Inc., New York.
14. Zhou, Y., An, W., Guo, F., et al. (2009) Principles of Electronic Warfare. Beijing: Publishing House of Electronics Industry, September 2009 (in Chinese).
15. Hu, L. (2004) Passive Location. Beijing: National Defence Industry Press, January 2004 (in Chinese).
16. Johnson, R.S., Cottrill, S.D. and Peebles, P.Z. (1969) A computation of radar SEP and CEP. IEEE Transactions on Aerospace and Electronic Systems, 3, 353-354.
17. Becker, K. (1996) A general approach to TMA observability from angle and frequency measurements. IEEE Transactions on Aerospace and Electronic Systems, 32 (1), 487-494.

## 3

## Single-Satellite Geolocation System Based on Direction Finding

Direction finding (DF) is the common way of electronic reconnaissance to get the direction of arrival (DOA), angle of arrival (AOA), or line-of-sight (LOS) information of the emitter signal. The geolocation system using LOS information is one of the most common systems applied in space electronic reconnaissance applications. Its basic principles are to take the first point of intersection of the LOS from the satellite to the emitter obtained from the DF system and the earth's surface as the location of the emitter. The advantages of the LOS geolocation method are a simple geolocation system, available single LOS geolocation, and fast geolocation. In addition, the emitter DOA, unlike other signal parameters such as carrier frequency and amplitude which, is instantaneously changing variable, is favorable to signal sorting. However, the disadvantages are that 2D DF equipment and attitude measurement equipment are required and the geolocation error is relatively large (especially in HEO).
Before discussing LOS geolocation technology, we first introduce several common electronic reconnaissance DF technologies, and then discuss single LOS geolocation technology and multitimes filtering technology for the satellite.

### 3.1 Direction Finding Techniques

The measurement of the DOA of an emitter from which the received radio signal was transmitted is called the radio direction finding or radio direction, direction finding (DF) for short. The basic technique of the electronic reconnaissance system for emitter DF is to use the amplitude or phase response of multiple antennas to judge from which direction the electromagnetic wave is coming. The DF system can be classified as an amplitude comparison DF system, phase comparison DF system, array DF system, rotational Doppler DF system, and other systems according to the DF techniques [1].


Figure 3.1 Theoretical diagram of a two-antenna amplitude comparison DF system: (a) beam configuration and (b) pattern of two-antenna system

### 3.1.1 Amplitude Comparison DF Technique

The amplitude comparison DF technique refers to the measurement of signal AOA through comparison of the relative amplitude of intercepted signals of different DF antennas or of the same DF antenna at different times [1].
In case there are two antennas A and B, suppose that the amplitude of a signal transmitted from the emitter to the reconnaissance antenna is $A(t)$, the gain of the receivers is $K_{A}$ and $K_{B}$, the pattern functions of two antennas nearest to the emitter are $F_{A}(\theta)$ and $F_{B}(\theta)$, respectively, and the axial directions of the two antennas are different, as shown in Figure 3.1. If the included angle between the axial direction of the two antennas is $\theta_{S}$, as shown in Figure 3.1, and the intersection angle is $\theta_{S} / 2$, the envelopes of the video frequency signal obtained after detection of the receiver circuit are

$$
\begin{align*}
& L_{A}=K_{A} A(t) F_{A}\left(\frac{\theta_{S}}{2}-\theta\right),  \tag{3.1}\\
& L_{B}=K_{B} A(t) F_{B}\left(\frac{\theta_{S}}{2}+\theta\right) . \tag{3.2}
\end{align*}
$$

After signals $L_{A}$ and $L_{B}$ go through the logarithmic amplifiers and a subtractor, then the output is

$$
\begin{align*}
Z & =\log L_{A}-\log L_{B} \\
& =\log \frac{L_{A}}{L_{B}} \\
& =\log \frac{K_{A}}{K_{B}}+\frac{F_{A}\left(\theta_{S} / 2-\theta\right)}{F_{B}\left(\theta_{S} / 2+\theta\right)} . \tag{3.3}
\end{align*}
$$

From expression (3.3) we can see that if $K_{A}=K_{B}$ and the pattern function of the receiving antennas $F_{A}(\cdot)$ and $F_{B}(\cdot)$ is known, $\theta$ can be solved from $Z$.
Different antennas have different pattern functions $F(\theta)$. The following is an analysis with $F(\theta)$ in the form of a Gaussian function as an example. This function can well approximate that of a wideband spiral antenna typically used in EW (electronic warfare) equipment, and the ideal antenna pattern function is

$$
\begin{equation*}
F_{A}(\theta)=F_{B}(\theta)=\exp \left[-\frac{K \theta^{2}}{\theta_{B}^{2}}\right] \tag{3.4}
\end{equation*}
$$

where $K$ is the proportional constant and $\theta_{B}$ is half of the half-power beam width (HPBW) of the antenna, that is, $\theta_{B}=1 / 2 \theta_{0.5}$.
Substitute expression (3.4) into expression (3.3) to obtain

$$
\begin{equation*}
Z=\log \frac{K_{A}}{K_{B}}+2 \theta_{S} \theta \frac{K \log e}{\theta_{B}^{2}} \tag{3.5}
\end{equation*}
$$

Then obtain

$$
\begin{equation*}
\theta=\frac{\theta_{B}^{2}}{2 \theta_{S} K \log e}\left(Z-\log \frac{K_{A}}{K_{B}}\right) \tag{3.6}
\end{equation*}
$$

If the channel gain is balanced, that is, $K_{A}=K_{B}$, expression (3.6) can be simplified to

$$
\begin{equation*}
\theta=\frac{\theta_{B}^{2}}{2 \theta_{S} K \log e} Z . \tag{3.7}
\end{equation*}
$$

Expression (3.7) is a relation of the final estimated value $\theta$, which shows that $\theta$ is directly proportional to $Z$. DF can be realized through measurement of the video envelope amplitude of two receiving channels. Since one pulse is the minimum required for DF, this method is also called the monopulse amplitude comparison DF technique.

### 3.1.2 Interferometer DF Technique

### 3.1.2.1 Theory of the Interferometer DF

The interferometer DF technique refers to the method of measuring the direction of incoming waves using the phase difference of antenna receipt signals in different wavefronts. Since this method is to obtain the DOA through comparison of the phases between two antennas, it is also called the phase comparison method. Theoretically, as the phase interferometer can also achieve monopulse DF, it is also called the phase monopulse DF. The simplest single baseline phase interferometer consists of two channels, as shown in Figure 3.2 [1, 2].
If an emitter is far away from the DF system, the incoming wave can be approximately seen as a plane wave. Suppose the plane wave is transmitted from the direction at an angle of $\theta$ with the antenna boresight and the transmitted narrowband signals received by two antennas are

$$
\begin{align*}
& s_{1}(t)=K_{1} \cos \left(2 \pi f t+\phi_{0}+\varphi\right),  \tag{3.8}\\
& s_{2}(t)=K_{2} \cos \left(2 \pi f t+\phi_{0}\right) . \tag{3.9}
\end{align*}
$$



Figure 3.2 Theoretical diagram of a single baseline phase interferometer

In expression (3.9), the phase difference $\phi$ between the two antennas caused by the wave path DOA of signals to the two antennas is

$$
\begin{equation*}
\phi=\frac{2 \pi l}{\lambda} \sin \theta . \tag{3.10}
\end{equation*}
$$

In expression (3.10), $\lambda$ is the signal wavelength and $l$ is the spacing between the two antennas. Let two signals go through the phase discriminator, which multiples two signals and conducts lowpass filtering. Then obtain the $U_{C}$ signal, conduct a $90^{\circ}$ phase shift of the one-channel signal, multiply by the other channel of the signal, carry out lowpass filtering and then obtain the $U_{S}$ signal. If two correlators have the same phase response, the signal correlated with the phase difference $\phi$ of the receiver output signal is

$$
\left.\begin{array}{l}
U_{C}=K \cos \phi  \tag{3.11}\\
U_{S}=K \sin \phi
\end{array}\right\}
$$

where $K=K_{1} K_{2} / 2$ is the system gain. According to expression (3.11), the following phase difference can be obtained:

$$
\begin{equation*}
\phi=\operatorname{arctg}\left(\frac{U_{S}}{U_{C}}\right) \tag{3.12}
\end{equation*}
$$

After the phase difference is measured, if the signal wavelength $\lambda$ and baseline length $l$ are known, according to expression (3.10), the DOA $\theta$ of the emitter signal can be obtained:

$$
\begin{equation*}
\theta=\arcsin \left(\frac{\phi \lambda}{2 \pi l}\right) . \tag{3.13}
\end{equation*}
$$

Expressions (3.12) and (3.13) can be established if and only if

$$
\begin{equation*}
|\phi \leq \pi|,|\theta| \leq \frac{\pi}{2} \tag{3.14}
\end{equation*}
$$

It is thus clear that the phase interferometer DF technique changes an estimation of the azimuth angle to an estimation of the phase difference $\phi$ caused by the path difference. The phase difference $\phi$ can be measured by quantizing the encoder in the digital signal processing (DSP) method. In addition, before estimating the phase difference $\phi$, the frequency or wavelength $\hat{\lambda}$ of the intercepted signal must be measured. That is to say, the phase interferometer DF (especially the single baseline DF ) must be supported by the measurement of frequency $f$.
The following section discusses special problems in the phase interferometer DF.

### 3.1.2.2 Phase Ambiguity Problem

As previously mentioned, the phase interferometer DF is used to estimate the emitter arrived angle $\theta$ by using the measured value of the phase difference $\phi$. The phase difference $\phi$ has a period of $2 \pi$. If the phase exceeds $2 \pi$, phase ambiguity may occur, thus preventing the true direction of the emitter to be found. The following introduces the derivation of the unambiguous visual angle $\theta_{u}$ of the phase interferometer.
The interferometer is boresight symmetric and can carry out DF on both sides. The maximum phase difference on one side of the boresight is $\pi$, the maximum phase difference on the other side is $-\pi$, and the single value range of $\phi$ is $[-\pi, \pi]$.
Substitute $\phi_{\max }=\pi$ into expression (3.13) and obtain

$$
\begin{equation*}
\theta_{\max }=\sin ^{-1}(\lambda / 2 l) \tag{3.15}
\end{equation*}
$$

Similarly, when $\phi_{\max }^{\prime}=-\pi$,

$$
\begin{equation*}
\theta_{\max }=-\sin ^{-1}(\lambda / 2 l) \tag{3.16}
\end{equation*}
$$

The unambiguous visual angle $\theta_{u}=\left|\theta_{\max }\right|+\left|\theta_{\max }^{\prime}\right|=2 \theta_{\max }$, that is,

$$
\begin{equation*}
\theta_{u}=2 \sin ^{-1}(\lambda / 2 l) . \tag{3.17}
\end{equation*}
$$

It is thus clear that to obtain a larger unambiguous angle area, a small antenna distance $l$ (short baseline) must be adopted. As for the interferometer whose antenna baseline length $l$ is determined, its unambiguous visual angle changes with the signal frequency. Additionally, from expression (3.17) we can obtain the maximum spacing between two antennas under which no ambiguity occurs:

$$
\begin{equation*}
l_{\max }=\frac{1}{2} \lambda . \tag{3.18}
\end{equation*}
$$

### 3.1.2.3 DF Accuracy Analysis

Due to the nonlinear transformation $\phi \rightarrow \theta$, the $\phi$ measurement error has obviously different effects on the $\theta$ estimation error when the angle $\theta$ is different. In order to find out the DF error source, by completely differentiating expression (3.10) one can obtain

$$
\begin{equation*}
\mathrm{d} \phi=\frac{2 \pi}{\lambda} l \cos \theta \mathrm{~d} \theta-\frac{2 \pi}{\lambda^{2}} l \sin \theta \mathrm{~d} \lambda+\frac{2 \pi}{\lambda} \sin \theta \mathrm{~d} l . \tag{3.19}
\end{equation*}
$$

As for two antennas with fixed length $l$, unstable factors of $l$ can be neglected (i.e., $\mathrm{d} l=0$ ) and the above expression can be simplified to

$$
\begin{equation*}
\mathrm{d} \phi=\frac{2 \pi}{\lambda} l \cos \theta \mathrm{~d} \theta-\frac{2 \pi}{\lambda^{2}} l \sin \theta \mathrm{~d} \lambda . \tag{3.20}
\end{equation*}
$$

Then obtain

$$
\begin{equation*}
\mathrm{d} \theta=\frac{\mathrm{d} \phi}{(2 \pi / \lambda) l \cos \theta}+\frac{\operatorname{tg} \theta}{\lambda} \mathrm{d} \lambda . \tag{3.21}
\end{equation*}
$$

Express the above expression by increment:

$$
\begin{equation*}
\Delta \theta=\frac{\Delta \phi}{(2 \pi / \lambda) l \cos \theta}+\frac{\Delta \lambda}{\lambda} \operatorname{tg} \theta . \tag{3.22}
\end{equation*}
$$

It can be seen from the above expression that:

1. The angle measurement error comes from the phase measurement error $\Delta \phi$ and the wavelength (or frequency) measurement error $\Delta \lambda$.
2. The angle measurement error value relates to the incoming angle $\theta$. When the azimuth angle coincides with the antenna axis $\left(\theta=0^{\circ}\right)$, the angle measurement error is minimum; when the azimuth angle coincides with the antenna baseline $\left(\theta=90^{\circ}\right)$, the angle measurement error is close to infinity and DF cannot be performed. Therefore, the incoming angle should not be too large, and in general should be limited to within $\pm 45^{\circ}$.
3. The angle measurement error also relates to the distance $l$ between two antennas. In order to obtain high angle measurement accuracy, $l$ must be long enough, that is to say, the long baseline interferometer (LBI) should be used. This conflicts with the unambiguous angle condition of the interferometer, which demands that $l$ be smaller than $\lambda / 2$.

In order to solve the conflict between high angle measurement accuracy and a large unambiguous visual angle scope, a multibaseline interferometer is commonly used.

### 3.1.2.4 Multibaseline Interferometer

As for a single-baseline interferometer, there are irreconcilable conflicts between improving DF accuracy and enlarging the scope of the visual angle. If the multibaseline interferometer is used, the conflict between the visual angle $\theta$ scope and angle measurement accuracy can be resolved. The interferometer with a shorter spacing will determine the visual angle scope and one with a longer spacing will determine angle measurement accuracy.
Figure 3.3 shows a one-dimensional three-baseline interferometer. Antenna ' 0 ' is the reference antenna, the spacing between antenna ' 1 ' and antenna ' 0 ' is $l_{1}$, the spacing between antenna ' 2 ' and antenna ' 0 ' is $l_{2}$, and the spacing between antenna ' 3 ' and antenna ' 0 ' is $l_{3}$. Suppose these antennas are omnidirectional antennas. If the emitter's plane wave arrives from the right side, the phase difference between each antenna and the reference antenna increases one by one from right to left. The unambiguous visual angle is [1-3]

$$
\begin{equation*}
\theta_{u}=2 \sin ^{-1}\left(\lambda / 2 l_{1}\right) . \tag{3.23}
\end{equation*}
$$



Figure 3.3 Principle diagram of a multibaseline interferometer

If the error caused by frequency variation is neglected, the angle measurement error is

$$
\begin{equation*}
\Delta \theta=\Delta \phi / 2 \pi\left(l_{3} / \lambda\right) \cos \theta \tag{3.24}
\end{equation*}
$$

Thus the multibaseline interferometer resolves the conflict between the visual angle range and angle measurement accuracy that exists in a single-baseline interferometer.
Suppose that the number of DF baselines of a one-dimensional multibaseline phase interferometer is $k$, the length ratio of adjacent baselines is $n$, and the angular quantization bits of the longest baseline encoder are $m$; in theory DF accuracy is

$$
\begin{equation*}
\delta \theta=\frac{\theta_{\max }}{n^{k-1} 2^{m-1}} \tag{3.25}
\end{equation*}
$$

The phase interferometer may provide higher DF accuracy, but it cannot cover all directions and has no simultaneous resolution to multiple signals. In addition, the phase difference relates to the signal frequency and frequency measurements are required during DF to obtain wavelength $\lambda$, which thus uniquely determine the DOA of signals.

### 3.1.2.5 2D Phase Interferometer

For a single-baseline interferometer in 3D space, the obtained angle is a cone angle, that is, the geolocation plane that can be determined is a conical surface with the interferometer baseline as its axial direction [1, 3], as shown in Figure 3.4.
In fact, the sight angle of the emitter in 3D space can be obtained through intersection of the azimuth and elevation planes. It is obvious that a one-dimensional interferometer is not enough


Figure 3.4 Equal-direction cosine cone of a single interferometer baseline


Figure 3.5 2D interferometer
to obtain azimuth and elevation 2D information. It is natural to wonder about the development of the interferometer from one dimension to two dimensions, thus measuring the angle both in the horizontal plane and the vertical plane.
As shown in Figure 3.5, a common 2D interferometer is composed of at least one pair of interferometers with mutually perpendicular baselines. The 2D angle information of this 2D interferometer includes

$$
\begin{align*}
& \varphi_{A}=\frac{2 \pi l_{A}}{\lambda} \sin \theta \cos \alpha,  \tag{3.26}\\
& \varphi_{B}=\frac{2 \pi l_{B}}{\lambda} \cos \theta \cos \alpha . \tag{3.27}
\end{align*}
$$

The following conclusions can be derived from the above two expressions:

1. When the elevation angle $\alpha \neq 0$, if approximating the emitter azimuth angle $\theta$ to the angle of a one-dimensional interferometer, an error factor $\cos \alpha$ will be introduced. If the elevation angle $\alpha$ is small, the corresponding error factor is small and negligible. For example, when the azimuth angle is $45^{\circ}$ and the elevation angle is $10^{\circ}$, the error caused by approximation is lower than $1^{\circ}$. However, when the elevation angle is larger, the corresponding error factor is larger and cannot be neglected. For example, when the azimuth angle and elevation angle
are both $45^{\circ}$, the error will reach $15^{\circ}$. In this case, the 2 D interferometer must be used to measure the azimuth angle $\theta$ and the elevation angle $\alpha$ of the emitter.
2. Estimation of the azimuth angle and the elevation angle:

$$
\begin{gather*}
\hat{\theta}=\operatorname{tg}^{-1}\left(\frac{l_{B} \phi_{A}}{l_{A} \phi_{B}}\right),  \tag{3.28}\\
\hat{\alpha}= \pm \cos ^{-1}\left(\frac{\lambda}{2 \pi} \sqrt{\frac{\phi_{A}^{2}+\phi_{B}^{2}}{l_{A}^{2} \sin ^{2} \theta+l_{B}^{2} \cos ^{2} \theta}}\right), \tag{3.29}
\end{gather*}
$$

where ' $\pm$ ' used in the above expression refers to the fact that the pointing of the antenna beam can be used to eliminate the ambiguity due to the interferometer's failure to distinguish up and down.

Like a one-dimensional interferometer, the 2D interferometer also has a conflict between the DF unambiguous angle area and DF accuracy. Thus, the 2D multibaseline interferometer should be used.

### 3.1.3 Array-Based DF Technique

The space spectrum estimation is a new DF technique combining a multiantenna array and modern digital signal processing (DSP) technology on the basis of a spectrum estimation [1].
For the convenience of discussing questions, a uniform linear array (ULA) is taken as an example, as shown in Figure 3.6. Suppose there are $M$ antennas in total and the distance between adjacent antenna elements is $d$; the time DOA at adjacent elements is

$$
\begin{equation*}
\tau=d \sin \theta / c \tag{3.30}
\end{equation*}
$$

where $\theta$ is the DOA and $c$ is the propagation speed of the radio wave in free space. By taking the signal received by the first element $s(t)$ as the reference, the output signal of the $m$ th element is

$$
\begin{equation*}
x_{m}(t)=s[t-(m-1) \tau]+n_{m}(t) \quad(m=1, \ldots, M) \tag{3.31}
\end{equation*}
$$

where $n_{m}(t)$ is the receiver noise, which is supposed to be neither correlated with the signal nor the noise of elements. From the above-mentioned expression it can be seen that signals received by elements are a time delay of the copy of an ideal undistorted signal of the first element $s(t)$ plus noise.
For a single sine wave signal, suppose that $s(t)=s_{0} \exp (j \omega t)$; the signal received by the $m$ th element is

$$
\begin{align*}
s[t-(m-1) \tau] & =s_{0} \exp \{j \omega[t-(m-1) \tau]\} \\
& =s_{0} \exp (j \omega t) \exp [-j 2 \pi d(m-1) \sin \theta / \lambda] \\
& =s(t) \exp [-j 2 \pi d(m-1) \sin \theta / \lambda] \quad(m=1, \ldots, M) \tag{3.32}
\end{align*}
$$

Let

$$
\begin{equation*}
f^{\prime}=d \sin \theta / \lambda \tag{3.33}
\end{equation*}
$$



Figure 3.6 Block diagram of a high-resolution array DF system

This can be seen as a 'spatial frequency,' which is correlated with the location and DOA of the incoming wave. Thus as for a ULA, the phase corresponding to the spatial frequency $f^{\prime}$ is

$$
\begin{equation*}
\phi_{m}=-2 \pi(m-1) f^{\prime} . \tag{3.34}
\end{equation*}
$$

This is a linear function of the spatial sample point and is equivalent to uniform sampling of time-domain signals. Thus, it can be seen from the above expression that, if the spectrum estimation method for time-domain signals is used to process the spatial sample point signal, the spatial frequency can also be estimated and the angle can be calculated according to expression (3.34). Thus the DF problem becomes a spatial spectrum estimation problem.

The above-mentioned derivation employs a ULA with spacing as $d$. If the nonuniform linear array (NULA) or other shaped arrays like the circular array and the square array are employed, a space spectrum estimation of direction can also be obtained but the method is different.
Therefore, the space spectrum estimation DF is used to estimate the spatial frequency according to the output signal of elements $\left\{X_{m}(t)\right\}$ and to determine other parameters. Among space spectrum estimation methods, the multiple signal classification (MUSIC) algorithm proposed by Schmidt in 1979 [4] is characterized by high accuracy, super-resolution, and so on, shows strong vitality, and has been widely used. Its basic principle is to correlate array output signal vectors, perform eigenvalue decomposition through a correlation matrix and obtain a space spectrum to identify multiple spatial signals and then search the DOA.
Compared with the traditional DF method, the space spectrum estimation method has outstanding advantages as follows:

1. High accuracy. DSP technology is used in array signals processing; thus various complicated mathematical tools can be used and, compared with traditional methods, the accuracy is much higher.
2. High resolution, which breaks through the Rayleigh limit and by which multiple signals falling into one beam can be resolved (thus, it is also called super-resolution DF).
3. Capability to perform DF of multiple signals arriving at the same time.
4. Capability to perform DF of a certain number of coherent sources and resolve direct signals and reflected signals under specific conditions.

The main shortcomings of this DF system are sensitivity to signal model distortion, larger computation load, and data size. Sensitivity is a difficulty in practical application. Large data size and computation load may affect its real-time capability. However, with the development of modern computer technology, these issues will finally be reasonably resolved and this system will has a very attractive application prospect.

### 3.1.4 Other DF Techniques

Besides monopulse amplitude comparison and phase comparison angle measurement methods, there are many other DF techniques, for example, the Adcock antenna DF technique commonly used in communication reconnaissance, the Wullenweber DF technique, the Doppler DF technique, and the multibeam DF techniques, like the multimode circular array multibeam DF and the lens feeder multibeam DF. Here we will not introduce them all. Readers interested in them may refer to references [1] and [5], and so on.

### 3.2 Single-Satellite LOS Geolocation Method and Analysis

A single-satellite LOS geolocation, that is, geolocation by azimuth angle and elevation angle of LOS, means that the satellite utilizes the LOS measured by DF and finds the intersection point with the earth's surface with LOS to determine the location of the emitter [6-10].

### 3.2.1 Model of LOS Geolocation

The satellite is operated in 3D space, so the azimuth angle and the elevation angle are required for LOS geolocation. Normally, the 2D interferometer is used to measure the azimuth angle and the elevation angle at each arrival signal from emitters on the earth's surface. Then the location of the emitter should be computed based on the satellite position (for example, longitude, latitude, and altitude) and attitude (yaw, pitch, roll angles) at that time.
As shown is Figure 3.7, it is assumed that the position of the satellite in the ECEF (earth-center earth-fixed) coordinates \{System $e\}$ is $\mathbf{x}_{s, e}$. Suppose that the longitude of the subsatellite point is $B_{s}$ and the latitude is $L_{s}$ (see Section 2.4.2). While the location of the emitter assumes $\mathbf{x}_{T, e}$, the vector of the distance between the emitter and the satellite in the ECEF coordinates should be

$$
\begin{equation*}
\mathbf{r}_{e}=\mathbf{x}_{S, e}-\mathbf{x}_{T, e} \tag{3.35}
\end{equation*}
$$

It is assumed that the coordinates could be rotated to measure the angle in the satellite body coordinates system. There is one way that $B_{s}$ is rotated by rotating axis $z$, as shown in Figure 3.7, and the $L_{s}$ is rotated upward by rotating axis $y$; then by reversing the coordinates transform, the LOS vector of the NED (north-east-down) coordinates system should be


Figure 3.7 Schematic figure of LOS geolocation
obtained. This is expressed mathematically as

$$
\begin{equation*}
\mathbf{r}_{n}=\mathbf{R}_{r g} \mathbf{R}_{y}\left(L_{s}\right) \mathbf{R}_{z}\left(B_{s}\right) \mathbf{r}_{e} \tag{3.36}
\end{equation*}
$$

where

$$
\mathbf{R}_{r g}=\left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{array}\right] .
$$

It is assumed that the attitude sensor outputs follow three values: (i) yaw angle $\psi$ : the included angle between the head of the satellite (axis $x$ in $\{$ system $b\}$ ) and the magnetic north (axis $x$ in \{system $n\}$ ), which is in the forward direction measured clockwise; (ii) pitch angle $\theta$ : the included angle between the right wing of the satellite (axis $y$ in $\{$ system $b\}$ ) and the east (axis $y$ in $\{$ system $n\}$ ), which is in the forward direction measured clockwise; and (iii) roll angle $\varphi$ : the included angle between the underbelly (axis $z$ in $\{$ system $b\}$ ) and the downward direction of the earth's center (axis $z$ in $\{$ system $n\}$ ), which is in the forward direction measured clockwise. A 3-2-1 Euler angle attitude rotation sequence is used, that is, a rotate roll angle horizontally along axis $x$, a rotate pitch angle along axis $y$, and then a rotate yaw angle along axis $z$. Based on that, we can obtain, in a satellite body coordinates system,

$$
\begin{equation*}
\mathbf{x}_{T, b}=\mathbf{M}\left(\mathbf{x}_{S, e}-\mathbf{x}_{T, e}\right), \tag{3.37}
\end{equation*}
$$

where $\mathbf{M}=\mathbf{R}_{x}^{\mathrm{T}}(\varphi) \mathbf{R}_{y}^{\mathrm{T}}(\theta) \mathbf{R}_{z}^{\mathrm{T}}(\psi) \mathbf{R}_{r g} \mathbf{R}_{y}\left(L_{s}\right) \mathbf{R}_{z}\left(B_{s}\right)$. In such a way, the correlation between the state and observation is established. The direction cosine angles of $\alpha$ and $\beta$ can be derived
from the expression of [8]

$$
\left[\begin{array}{l}
\alpha  \tag{3.38}\\
\beta
\end{array}\right]=\left[\begin{array}{c}
\arccos \frac{x_{T, b}}{\sqrt{x_{T, b}^{2}+y_{T, b}^{2}+z_{T, b}^{2}}} \\
\arccos \frac{y_{T, b}}{\sqrt{x_{T, b}^{2}+y_{T, b}^{2}+z_{T, b}^{2}}}
\end{array}\right]=\mathbf{f}\left(\mathbf{x}_{T, e}\right) .
$$

If we use the WGS-84 mean earth ellipsoid model, and the prior knowledge that the emitter is on the earth's surface, the equivalent form for the terrestrial longitude and latitude ( $L_{T}, B_{T}$ ) and the terrestrial altitude $\left(H_{T}\right)$ on earth and its coordinate $\mathbf{X}_{T, e}=\left[\begin{array}{lll}x_{T, e} & y_{T, e} & z_{T, e}\end{array}\right]^{\mathrm{T}}$ in the ECEF system is that

$$
\left.\begin{array}{l}
x_{T, e}=\left(N+H_{T}\right) \cos B_{T} \cos L_{T} \\
y_{T, e}=\left(N+H_{T}\right) \cos B_{T} \sin L_{T}  \tag{3.39}\\
z_{T, e}=\left(N\left(1-e^{2}\right)+H_{T}\right) \sin B_{T}
\end{array}\right\}
$$

where $N=a / \sqrt{1-e^{2} \sin ^{2} B_{T}}$ is the local radius of curvature in a prime vertical circle. The radius of the earth $a=6378137 \mathrm{~m}$. The square of the first eccentricity $e^{2}=0.006694379$ 99013.

Expression (3.39) can be written in another form as

$$
\begin{equation*}
\frac{x_{T, e}^{2}+y_{T, e}^{2}}{(N+H)^{2}}+\frac{z_{T, e}^{2}}{\left[N\left(1-e^{2}\right)+H\right]^{2}}=1 \tag{3.40}
\end{equation*}
$$

Combined with the measurement expressions (3.38) and (3.40), there are three unknown values in the three expressions, and by eliminating one result with a longer distance from the other point on the earth, the location of the emitter can be calculated.

### 3.2.2 Solution of LOS Geolocation

Here we will discuss the solution of this expression. If the distance between the emitter and satellite is $r$, after $\alpha$ and $\beta$ are measured, the direction cosine angle measured in the satellite body coordinates system can be decomposed as

$$
\begin{equation*}
\mathbf{x}_{T, b}=r \mathbf{u} \tag{3.41}
\end{equation*}
$$

where

$$
\mathbf{u}=\left[\begin{array}{c}
\cos \alpha \\
\cos \beta \\
\sqrt{1-\cos ^{2} \alpha-\cos ^{2} \beta}
\end{array}\right]
$$

After expression (3.41) is substituted into expression (3.37), we can obtain

$$
\begin{equation*}
\mathbf{x}_{T, e}=\mathbf{x}_{S, e}-\mathbf{M}^{-1} \mathbf{x}_{T, b}=\mathbf{x}_{S, e}-r \mathbf{M}^{-1} \mathbf{u} \tag{3.42}
\end{equation*}
$$

A quadratic expression with one unknown $r$ can be obtained by substituting expression (3.42) into expression (3.40). Two roots of $\widehat{r}_{1}$ and $\widehat{r}_{2}$ can be solved using this expression. The minimum one is the distance value of the emitter. By substituting the solved distance value into expression (3.42), the location of the emitter is measured.
In the solution of expression (3.40), the local radius of curvature $N$ of the emitter in a prime vertical circle is unknown, so the iterative approach should be used for solution. By considering that the $N$ value of the satellite is close to the $N$ value of the emitter, firstly the value of $N_{0}=$ $N_{\text {Sat }}$ is substituted into expression (3.40) to calculate the iterative distance $r_{i}$ of the $i$ th time, and then the location of the emitter is measured by expression (3.42). Thus the local radius of curvature $N_{i}$ in a prime vertical circle estimated by the $i$ th iteration is measured, and so on, until the distance of the emitter converges:

$$
\left|r_{i}-r_{i-1}\right| \leq \epsilon_{r}
$$

where $\epsilon_{r}$ is the set threshold of the small distance error.

### 3.2.3 CRLB of the LOS Geolocation Error

Based on the analysis above, the factors that affect the accuracy of geolocation include the direction cosine measurement error, attitude measurement error, altitude error of emitter, position error of the satellite, and so on. The main factors are the measurement error of the direction cosine and the altitude error of the emitter. Under the two conditions, the Cramér-Rao lower bound (CRLB) of the geolocation error can be analyzed.
To get the CRLB of LOS geolocation, it is assumed that all these errors are joint Gaussian distributions with zero mean, where the $\mathbf{b}_{\alpha \beta H}^{m}$ denotes the measured values of $\alpha, \beta$, and the assumed altitude $H$. This is equal to

$$
\mathbf{b}_{\alpha \beta H}^{m}=\mathbf{b}_{\alpha \beta H}\left(\mathbf{x}_{B L H}\right)+\mathbf{n}_{\alpha \beta H},
$$

where the true value of $\mathbf{b}_{\alpha \beta H}=[\alpha \beta H]^{\mathrm{T}}$ is the function of the emitter position $\mathbf{x}_{B L H}$, and $\mathbf{n}_{\alpha \beta H}$ is the error vector of the measured values of $\alpha, \beta$, and $H$. By assuming $E\left[\mathbf{n}_{\alpha \beta H}\right]=\mathbf{0}$ and $E\left[\mathbf{n}_{\alpha \beta H} \mathbf{n}_{\alpha \beta H}^{\mathrm{T}}\right]=\mathbf{Q}_{\alpha \beta H}$, the probability density function of the measurement error is

$$
\begin{align*}
f\left(\mathbf{b}_{\alpha \beta H} \mid \mathbf{x}_{B L H}\right)= & \frac{1}{\left|2 \pi \mathbf{Q}_{\alpha \beta H}\right|^{1 / 2}} \\
& \exp \left\{-\left[\mathbf{b}_{\alpha \beta H}^{m}-\mathbf{b}_{\alpha \beta H}\left(\mathbf{x}_{B L H}\right)\right]^{\mathrm{T}} \mathbf{Q}_{\alpha \beta H}^{-1}\left[\mathbf{b}_{\alpha \beta H}^{m}-\mathbf{b}_{\alpha \beta H}\left(\mathbf{x}_{B L H}\right)\right] / 2\right\} . \tag{3.43}
\end{align*}
$$

Based on the definition of the Fisher information matrix [11], by differentiating Equation (3.43) in matrix form, we can obtain

$$
\begin{align*}
\mathbf{F} & =-\mathbf{E}\left[\frac{\partial^{2} \ln f\left(\mathbf{b}_{\alpha \beta H} \mid \mathbf{x}_{B L H}\right)}{\partial \mathbf{x}_{B L H}^{2}}\right], \\
& =\mathbf{J}^{T} \mathbf{Q}_{\alpha \beta H}^{-1} \mathbf{J} \tag{3.44}
\end{align*}
$$

where

$$
\mathbf{J}=\left[\begin{array}{ccc}
\frac{\partial \alpha}{\partial B} & \frac{\partial \alpha}{\partial L} & \frac{\partial \alpha}{\partial H} \\
\frac{\partial \beta}{\partial B} & \frac{\partial \beta}{\partial L} & \frac{\partial \beta}{\partial H} \\
0 & 0 & 1
\end{array}\right]
$$

is the Jacobian matrix, and the first two items of the last row are 0 because the longitude $L$, latitude $B$, and altitude $H$ are orthogonal.
The solution of the Jacobian matrix J in expression (3.44) is described below. Firstly, by solving the total differential according to Equation 3.38, we obtain

$$
\left[\begin{array}{c}
\mathrm{d} \alpha  \tag{3.45}\\
\mathrm{~d} \beta
\end{array}\right]=\mathbf{J}_{1} \mathrm{~d} \mathbf{X}_{T, b},
$$

where the Jacobian matrix is

$$
\mathbf{J}_{1}=\left[\begin{array}{ccc}
\frac{\partial \alpha}{\partial x_{T, b}} & \frac{\partial \alpha}{\partial y_{T, b}} & \frac{\partial \alpha}{\partial z_{T, b}} \\
\frac{\partial \beta}{\partial x_{T, b}} & \frac{\partial \beta}{\partial y_{T, b}} & \frac{\partial \beta}{\partial z_{T, b}}
\end{array}\right] .
$$

By solving the total differential to expression (3.37),

$$
\begin{equation*}
\mathrm{d} \mathbf{x}_{T, b}=\mathbf{M} \mathrm{d} \mathbf{x}_{T, e} . \tag{3.46}
\end{equation*}
$$

By solving the total differential to expression (3.39),

$$
\begin{equation*}
\mathrm{d} \mathbf{x}_{T, e}=\mathbf{J}_{2} \mathrm{~d} \mathbf{x}_{B L H} \tag{3.47}
\end{equation*}
$$

where the Jacobian matrix is

$$
\mathbf{J}_{2}=\left[\begin{array}{ccc}
\frac{\partial x_{T, e}}{\partial B} & \frac{\partial x_{T, e}}{\partial L} & \frac{\partial x_{T, e}}{\partial H} \\
\frac{\partial y_{T, e}}{\partial B} & \frac{\partial y_{T, e}}{\partial L} & \frac{\partial x_{T, e}}{\partial H} \\
\frac{\partial z_{T, e}}{\partial B} & \frac{\partial z_{T, e}}{\partial L} & \frac{\partial z_{T, e}}{\partial H}
\end{array}\right]
$$

and

$$
\mathrm{d} \mathbf{x}_{B L H}=\left[\begin{array}{lll}
\mathrm{d} B & \mathrm{~d} L \mathrm{~d} H
\end{array}\right]^{T}
$$

Combined with Equations (3.45) to (3.47), we can obtain

$$
\left[\begin{array}{c}
\mathrm{d} \alpha  \tag{3.48}\\
\mathrm{~d} \beta
\end{array}\right]=\mathbf{J}_{3} \mathrm{~d} \mathbf{x}_{B L H}
$$

where $\mathbf{J}_{3}=\mathbf{J}_{1} \mathbf{M} \mathbf{J}_{2}$ is a $2 \times 3$ matrix. Based on the orthogonality between latitude $B$, longitude $L$, and altitude $H$, we can obtain

$$
\mathbf{J}=\left[\begin{array}{ll}
\mathbf{J}_{3}^{\mathrm{T}} & \mathbf{l} \tag{3.49}
\end{array}\right]^{\mathrm{T}},
$$

where $\mathbf{I}=\left[\begin{array}{lll}0 & 0 & 1\end{array}\right]^{\mathrm{T}}$. After expression (3.49) is substituted into expression (3.44), the CRLB of LOS geolocation for the geodetic coordinate system is

$$
\begin{equation*}
\mathbf{C R L B}_{B L H}=\left(\mathbf{J}^{\mathrm{T}} \mathbf{Q}_{\alpha \beta H}^{-1} \mathbf{J}\right)^{-1} \tag{3.50}
\end{equation*}
$$

where $\mathbf{Q}_{\alpha \beta H}$ is the covariance matrix of the direction cosine angle and altitude error.
The lower bound of the geolocation error for the emitter angle is determined by this covariance matrix:

$$
\begin{equation*}
\sigma_{p_{\min }}=\left\{N_{t}^{2}\left[\mathbf{C R L B}_{B L H}(1,1)+\mathbf{C R L B}_{B L H}(2,2)\right]+\mathbf{C R L B}_{B L H}(3,3)\right\}^{1 / 2} \tag{3.51}
\end{equation*}
$$

where $N_{t}$ is the radius of curvature in the local prime vertical circle.

### 3.2.4 Simulation and Analysis of the LOS Geolocation Error

Suppose the altitude of satellite $H_{s}=500 \mathrm{~km}$, the longitude and latitude of the subsatellite point is $\left(L_{s}, B_{s}\right)=\left(40^{\circ}, 120^{\circ}\right)$, the longitude and latitude of the emitter lies in $\left(L_{t}, B_{t}\right)=\left(35^{\circ}, 116^{\circ}\right)$, and the attitude angles of the satellite (yaw angle, pitch angle, roll angle) are equal to ( $45^{\circ}, 0^{\circ}$, $0^{\circ}$ ). If the measurement errors of direction cosine $\alpha$ and $\beta$ are zero-mean Gaussian white noise and uncorrelated with each other, then the angle root mean square error (RMSE) is $\sigma_{\alpha}=\sigma_{\beta}$, by assuming that the RMSE of the ground altitude is $\mathrm{d} H$.

When there is an error in the angle measurement, based on the geolocation solution in Section 3.2.2, the estimated value for geolocation $\mathbf{x}_{T, e}$ in ECEF coordinates at any time can still be solved, which is converted to an estimated value of the emitter location in the geodetic coordinates system ( $\widehat{B}_{t}, \widehat{L}_{t}, \widehat{H}_{t}$ ). To reduce the effects of random errors, the RMSE for the locating position should be counted by repeating the Monte Carlo test (for example, $M=5000$ times):

$$
\begin{equation*}
\sigma_{p}=\left(\frac{1}{M} \sum_{m=1}^{M}\left(\widehat{B}_{t}-B_{t}\right)^{2} \widehat{N}_{t}^{2}+\left(\widehat{L}_{t}-L_{t}\right)^{2} \widehat{N}_{t}^{2}+\left(\widehat{H}_{t}-H_{t}\right)^{2}\right)^{1 / 2} \tag{3.52}
\end{equation*}
$$

When the error of the emitter altitude $\mathrm{d} H=10 \mathrm{~m}$ (with respect to ships in the sea), the relation between RMSE $\sigma_{p}$ and the accuracy of the measured angle under different error conditions for LOS geolocation is stated as shown in Figure 3.8.
As shown in Figure 3.8, in most cases, the results solved by the geolocation solution in Section 3.2.2 reach to CRLB. However, if the angle error exceeds $2^{\circ}$, the actual geolocation error is slightly more than the theoretical CRLB of 1 dB . This shows that the method described in Section 3.2.2 is the optimal geolocation method if the measured angle error is small, which approaches the optimal geolocation method if the measured angle error is large.
In Figure 3.8, the geolocation error is exponentially increased over the angle measurement error. Therefore, the error for the angle measurement should be small enough to minimize the geolocation error.
When the error of the ground altitude $\mathrm{d} H=1000 \mathrm{~m}$ (with respect to rugged mountain areas in the land), the relation between the geolocation error and the different accuracy of the measured angle is stated as shown in Figure 3.9.
As shown in Figure 3.9, since there was an error in the emitter altitude assumption, the geolocation error does not decrease while the error of angle measurement decreases, but is stable
at a level where the actual error is higher than the emitter altitude error (approximately two times). This shows that the emitter altitude assumption error affects not only for LOS geolocation itself but also an additional error for LOS geolocation. Consequently, the altitude error should be eliminated in LOS geolocation. The digital altitude map can be used for eliminating the geolocation error; that is, firstly the digital altitude at this point is coarsely measured by the estimated emitter location, followed by a modified estimation of the emitter location, and then by iteration for a couple of times, until the effect of the altitude error is small enough.

### 3.2.5 Geometric Distribution of the LOS Geolocation Error

From LOS geolocation research, we should be concerned about the distribution of LOS geolocation error on the earth, especially its distribution near the subsatellite point. Intuitively, the further from the subsatellite point, the smaller is the elevation angle is, as also is the geolocation error. The distribution of the geolocation error seems to be in a concentric circle around the subsatellite point. We will verify this in this section.
If the altitude of satellite $H_{s}=500 \mathrm{~km}$, the longitude and latitude of the subsatellite point should be $\left(L_{s}, B_{s}\right)=\left(40^{\circ}, 120^{\circ}\right)$. By assuming that the LOS angle accuracy is $0.1^{\circ}$, the attitude angles of the satellite (heading angle, pitch angle, roll angle) are equal to $\left(40^{\circ}, 0.1^{\circ}, 2.1^{\circ}\right)$ and the prior emitter altitude assumption error is 1000 m . As a result, the GDOP (geometric dilution of precision) distribution of the ground geolocation error is shown in Figure 3.10.
As shown in Figure 3.10, the minimum geolocation error is found at the subsatellite point. The further from the subsatellite point, the greater is the error. This shows the same as our intuitive sense. If observed carefully, just as the direction cosine angle features nonlinear performance, the GDOP distribution graph made by the attitude angles has a certain rotation.


Figure 3.8 Relation between a LOS geolocation error and an angle error (altitude error $\mathrm{d} H=10 \mathrm{~m}$ )


Figure 3.9 Relation between a LOS geolocation error and an angle error (altitude error $\mathrm{d} H=1 \mathrm{~km}$ )

Because of the linear nature of the direction cosine and attitude transformation, as a result the distribution of the geolocation error is derived from the concentric circle at the subsatellite point.
If we decompose $\mathbf{C R L B}_{\text {BLH }}$ using the eigenvalue and the eigenvector, correspondingly the major axis and the minor axis of the error ellipse may be measured. Readers could make an analysis if you are interested, so we will leave this problem to readers.

### 3.3 Multitimes Statistic LOS Geolocation

In actual satellite reconnaissance operation, signals received from the same emitter would occur more than once. If we process the measured data in multiple measurements, a better result than that of single-time LOS geolocation may be obtained.
Under multitimes angle measurement, there are various ways to locate the fixed emitter using a single satellite:

1. By using the intersected LOS lines through multitimes DF measurement of the satellite at different orbit locations to measure the location of the target emitter. This method is called multitimes triangulation (or bearings-only localization).
2. Single-satellite LOS geolocation is used for a fixed emitter. Average processing is carried out based on the point of intersection with the earth's surface each time, in order to improve the accuracy of geolocation.
3. Single-satellite LOS geolocation is used for a fixed emitter. Weighted average processing is carried out based on the point of intersection with the earth's surface each time, in order to improve the accuracy of geolocation.

We will analyze the three methods respectively in the following context.


Figure 3.10 GDOP contour map for a LOS geolocation error

### 3.3.1 Single-Satellite Multitimes Triangulation

Triangulation is referred to the emitter geolocation by the intersected points of direction lines via multitimes DF measurements for the same emitter.
If we use the ECEF coordinates system, the state vector of the emitter position is $\mathbf{X}_{k}=$ $\left[x_{t}(k), y_{t}(k), z_{t}(k)\right]^{\mathrm{T}}$. As the emitter is considered as a fixed emitter on earth, we can obtain the state equation [12]:

$$
\begin{equation*}
\mathbf{X}_{k}=\mathbf{X}_{k-1} . \tag{3.53}
\end{equation*}
$$

If a 2D interferometer is used for measuring the LOS angle of an incoming wave of the emitter, the angles may be transformed to an azimuth angle $\beta(k)$ and an elevation angle $\epsilon(k)$ in the ECEF coordinates system based on the attitude parameter of the satellite. The correlation between the angles and the state is expressed as

$$
\begin{align*}
& \beta_{m}(k)=\operatorname{arctg} \frac{x_{s}(k)-x_{t}(k)}{y_{s}(k)-y_{t}(k)}+v_{\beta}(k),  \tag{3.54}\\
& \epsilon_{m}(k)=\operatorname{arctg} \frac{z_{s}(k)-z_{t}(k)}{\sqrt{\left[x_{s}(k)-x_{t}(k)\right]^{2}+\left[y_{s}(k)-y_{t}(k)\right]^{2}}}+v_{\epsilon}(k) . \tag{3.55}
\end{align*}
$$

Let the observation vector $\mathbf{h}_{k}=\left[\beta_{m}(k) \epsilon_{m}(k)\right]^{\mathrm{T}}$. The extended Kalman filter (EKF) algorithm can be used for estimating the position of the emitter $\mathbf{X}_{k}$. In the EKF, the measurement
equation can be linearized at the predicted state vector $\widehat{\mathbf{X}}_{k / k-1}$. The Jacobian matrix is computed by a series of derivative operations:

$$
\begin{equation*}
\mathbf{H}_{k}=\left.\frac{\partial \mathbf{h}_{k}}{\partial \mathbf{X}}\right|_{\mathbf{X}=\hat{\mathbf{x}}_{k / k-1}} \tag{3.56}
\end{equation*}
$$

The expanded EKF algorithm is shown by the following equations [12]:

$$
\begin{align*}
\widehat{\mathbf{X}}_{k / k-1} & =\widehat{\mathbf{X}}_{k-1 / k-1},  \tag{3.57}\\
\mathbf{P}_{k / k-1} & =\mathbf{P}_{k-1 / k-1},  \tag{3.58}\\
\mathbf{K}_{k} & =\mathbf{P}_{k / k-1} \mathbf{H}_{k}^{\mathrm{T}}\left(\mathbf{H}_{k} \mathbf{P}_{k / k-1} \mathbf{H}_{k}^{\mathrm{T}}+\mathbf{R}_{k}\right)^{-1},  \tag{3.59}\\
\widehat{\mathbf{X}}_{k / k} & =\widehat{\mathbf{X}}_{k / k-1}+\mathbf{K}_{k}\left[\mathbf{Z}_{k}-\mathbf{h}\left(\widehat{\mathbf{X}}_{k / k-1}\right)\right],  \tag{3.60}\\
\mathbf{P}_{k / k} & =\left(\mathbf{I}-\mathbf{K}_{k} \mathbf{H}_{k}\right) \mathbf{P}_{k / k-1}\left(\mathbf{I}-\mathbf{K}_{k} \mathbf{H}_{k}\right)^{\mathrm{T}}+\mathbf{K}_{k} \mathbf{R}_{k} \mathbf{K}_{k}^{\mathrm{T}}, \tag{3.61}
\end{align*}
$$

where the initial value $\widehat{\mathbf{X}}_{0}$ and the initial variance $\mathbf{P}_{0}$ are the single-time instantaneous geolocation value and the corresponding error variance, respectively.
When the EKF algorithm is used for multitimes geolocation, due to the assumption that the target emitter on the earth's surface is not used, the emitter with an unknown altitude may be located, that is, without prior information of the emitter altitude. Because of that, the geolocation error is great when the angle error is large, so we need many more observation times and observed points. Therefore, this algorithm is suitable for the geolocation of a fixed emitter under a long observation time, a large angle of satellite track, or high angle accuracy.

### 3.3.2 Average for Single-Satellite Multitimes Geolocation

For single-satellite LOS geolocation, the location of the emitter can be computed via the intersecting point of the LOS with the earth's surface. If the geolocation results at each time are filtered statistically, the accuracy of geolocation may be improved. Since the location of the target emitter is fixed in the ECEF coordinates system, the statistical averaging for the results could be carried out to estimate the location of the target emitter.
By assuming that the estimated value of the emitter location by instantaneous geolocation at time $k$ is $\widehat{\mathbf{X}}_{k}$, we can obtain the estimated geolocation value after $K$ angles:

$$
\begin{equation*}
\widehat{\mathbf{X}}=\frac{1}{K} \sum_{k=1}^{K} \widehat{\mathbf{X}}_{k} \tag{3.62}
\end{equation*}
$$

Due to the satellite position at different measurement times being different, the geolocation error (or uncertainty area) after intersecting with the earth's surface is also not the same. If we use the simple averaging method above, the data with many errors may 'pollute' the geolocation data with high accuracy. To avoid this, the range of angles between the emitter and the subsatellite point is used to weigh the geolocation results, in order to improve the accuracy of geolocation.

### 3.3.3 Weighted Average for Single-Satellite Multitimes Geolocation

If the emitter location calculated by LOS geolocation at time $k$ is $\widehat{\mathbf{X}}_{k}$, correspondingly the measurement error $\mathbf{e}_{k}$ is zero-mean and the covariance matrix is $\mathbf{R}_{k}$. By assuming that two measurements at any two times are uncorrelated, after the angle of $K$ times is measured, the expression may be written as

$$
\begin{equation*}
\widehat{\mathbf{X}}^{K}=\mathbf{H}^{K} \mathbf{X}+\mathbf{e}^{K} \tag{3.63}
\end{equation*}
$$

where $\widehat{\mathbf{X}}^{K}=\left[\begin{array}{lll}\mathbf{X}_{1} & \cdots & \mathbf{X}_{K}\end{array}\right]^{\mathrm{T}}, \mathbf{H}^{K}=\left[\begin{array}{lll}\mathbf{I}_{3} & \cdots & \mathbf{I}_{3}\end{array}\right]^{\mathrm{T}}$, and $\mathbf{e}^{K}=\left[\begin{array}{lll}\mathbf{e}_{1} & \cdots & \mathbf{e}_{K}\end{array}\right]^{\mathrm{T}}$. Based on the weighted least squares, the optimal weighted matrix corresponding to the expression (3.63) should be

$$
\mathbf{W}=\left\{\mathbf{E}\left[\mathbf{e}^{K}\left(\mathbf{e}^{K}\right)^{\mathrm{T}}\right]\right\}^{-1}=\left[\begin{array}{lll}
\mathbf{R}_{1}^{-1} & & \\
& \cdots & \\
& & \mathbf{R}_{K}^{-1}
\end{array}\right]
$$

By the weighted least squares (WLS) method, the estimated location is the best linear unbiased estimation (BLUE), which is [11, 12]

$$
\begin{equation*}
\widehat{\mathbf{X}}=\left(\left(\mathbf{H}^{K}\right)^{\mathrm{T}} \mathbf{W} \mathbf{H}^{K}\right)^{-1}\left(\mathbf{H}^{K}\right)^{\mathrm{T}} \mathbf{W} \mathbf{X}^{K}=\left(\sum_{k=1}^{K} \mathbf{R}_{k}^{-1}\right)^{-1} \sum_{k=1}^{K} \mathbf{R}_{k}^{-1} \widehat{\mathbf{X}}_{k} \tag{3.64}
\end{equation*}
$$

Since the $\widehat{\mathbf{X}}_{k}$ is an estimated value in Section 3.2.2, the true value of the emitter is unknown, so the corresponding error matrix $\mathbf{R}_{k}$ is not able to be measured. However, an approximation can be made. Based on the estimated location $\widehat{\mathbf{X}}_{k}$, the $\mathbf{C R L B}_{B L H}$ of the emitter geolocation error is measured using the algorithm in Section 3.2.3 and the approximate variance matrix $\mathbf{R}_{k}$ of the emitter geolocation error is substituted into Equation 3.64 to give the weighted average.

### 3.3.4 Simulation of Single-Satellite LOS Geolocation

When the effect of the random angle measurement error on the LOS emitter geolocation accuracy is evaluated, to eliminate the influence from random noises in single-satellite LOS geolocation, the Monte-Carlo method is usually used to carry out repeated tests in calculating the average geolocation error statistically. The Monte-Carlo method needs on average at least 50 times or more simulations.
Assume that the conditions for the simulation scenario is as follows: the satellite orbital altitude is 600 km , the circular orbit has an inclination of $45^{\circ}$, orbit elements ( $6978 \mathrm{~km}, 0$, $45^{\circ}, 0,0,0$ ), the target emitter is located at Taipei and the method is a single-satellite LOS geolocation.
By using the STK (satellite tool kit) software, we can acquire data of the satellite operating process, such as the satellite access area and access time for the emitter. The access area is shown in Figure 3.11.
The satellite repeated access interval and access duration within a day are shown in Table 3.1.


Figure 3.11 Coverage information on a subsatellite track

Table 3.1 Statistics of the satellite repeated access interval and access duration

| Times | Start time | End time | Duration (s) |
| :--- | :--- | :--- | :---: |
| 1 | 1 January 2001 09:40:09.26 | 1 January 2001 09:52:27.26 | 737.994 |
| 2 | 1 January 2001 11:20:51.09 | 1 January 2001 11:34:08.46 | 797.380 |
| 3 | 1 January 2001 13:04:28.31 | 1 January 2001 13:14:56.93 | 628.622 |
| 4 | 1 January 2001 14:49:01.89 | 1 January 2001 14:56:51.62 | 469.724 |
| 5 | 1 January 2001 16:31:05.18 | 1 January 2001 16:41:21.02 | 615.840 |
| 6 | 1 January 2001 18:11:53.76 | 1 January 2001 18:25:06.11 | 792.353 |
| 7 | 1 January 2001 19:53:28.67 | 1 January 2001 20:05:59.83 | 751.151 |

Choose satellite data of the coverage track from 11:20:51.09 on 1 January 2001 to 11:34:08.46 on 1 January 2001. Suppose the data rate is $T=10$ seconds one time; then we get the measured angle changing curve shown in Figure 3.12:

1. Simulation of single-satellite triangulation performance

Carry out the Monte-Carlo test for 100 times with different levels of angle measurement error. We can obtain the geolocation error of the triangulation method as shown in Figure 3.13 by using the recursive nonlinear least squares method mentioned in Section 3.3.1.

It is known from Figure 3.13 that the geolocation error decreases along with the angle error, but the angle error changes from $3^{\circ}$ to $0.5^{\circ}$, and the final geolocation convergence error is almost the same, approximately within $8-11 \mathrm{~km}$. However, when the angle measurement error decreases from $0.5^{\circ}$ to $0.1^{\circ}$, the geolocation error decreases comparatively as dramatically and the triangulation accuracy can be within 1 km .


Figure 3.12 Angle curves
2. Simulation of the single-satellite instantaneous angle measurement geolocation filter method

If the single-satellite LOS geolocation method is adopted, after the satellite starts to enter the satellite visible region, as the elevation angle is very large when the satellite is just entering or leaving the coverage area of emitter illumination, the LOS and earth's surface are close to being tangent, which leads to a large geolocation error. When the satellite passes the zenith of the emitter or reaches the subsatellite point, the geolocation error is the smallest. Assume that the DF error is $\sigma_{\beta}=\sigma_{\epsilon}=1^{\circ}$ and $T=10$ seconds. Then changing the diagram of the satellite theoretical geolocation error during one time coverage period can be drawn as shown in Figure 3.14.

It is known from Figure 3.14 that the geolocation error is huge when the satellite is just entering or leaving the coverage area, which is up to tens of thousands of kilometers. Therefore, this part of the data may impose a bad effect on the average situation of geolocation statistics.

Suppose the LOS measurement error $\sigma_{\beta}=\sigma_{\epsilon}=1^{\circ}$. With the above simulation scenario and data we can get a discrete distribution on the plane of geolocation points during one time satellite passing at the zenith, as shown in Figure 3.15, where ' $\oplus$ ' in the figure represents the actual position of the emitter.


Figure 3.13 Convergence curve of a geolocation error

If an average method is adopted, by using average geolocation points we can acquire the distribution of geolocation points as shown in Figure 3.16. It is clear that the distribution area of geolocation points narrows, showing the reduction in geolocation errors. However, it is easy to find that the distribution center of the average deviates from the actual position of the emitter. This is because a kind of bias exists in LOS geolocation.

If a weighted average is used, the distribution of geolocation points is as shown in Figure 3.17.

By comparing Figure 3.17 with Figure 3.16, we can see that the distribution area shrinks further after taking a weighted average. However, the distribution center still deviates from the real position of the emitter.

Repeat the Monte-Carlo trials 100 times. We can record the geolocation error under conditions of different angle measurement errors, as shown in Table 3.2.

From Table 3.2 we can see that bias and standard (STD) error of the average method decrease along with angle error. However, in using the weighted average geolocation method, bias does not decrease while STD error does. Therefore, we must considering how to eliminate bias when using the weighted average method. Also, when the case angle measurement error is large, adoption of the weighted average method can reduce the CEP (circular error probability) and STD error of the geolocation error.
3. Conclusion

By comparing the triangulation method, average method, and weighted average method, we know that the triangulation method, though it does not use prior information of the earth's surface, can reach comparatively high geolocation accuracy when the angle


Figure 3.14 Changing curve of a theoretical geolocation error during the process of the satellite one time passing of zenith


Figure 3.15 Distribution of geolocation points before average
measurement error is small. The average method is not as good as the weighted average geolocation method when the angle measurement error is large. However, there is bias in the weighted average method, which imposes influence on geolocation that cannot be neglected and can only be solved by using some effort.


Figure 3.16 Distribution of geolocation points after average


Figure 3.17 Distribution of geolocation points after weighted average

Table 3.2 Statistics of the geolocation error under the condition of different angle measurement errors

| Angle error | Geolocation error of average (km) |  |  | Geolocation error of weighted average (km) |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | CEP | Bias | STD | CEP | Bias | STD |
| $3^{\circ}$ | 30 | 8.4 | 39 | 19 | 14 | 14 |
| $1^{\circ}$ | 15 | 3.5 | 22 | 10 | 9.6 | 5.2 |
| $0.5^{\circ}$ | 10.5 | 2.1 | 15 | 9.7 | 9.6 | 2.4 |
| $0.1^{\circ}$ | 3.8 | 0.8 | 5.3 | 9.6 | 9.6 | 0.55 |

### 3.4 Single HEO Satellite LOS Geolocation

### 3.4.1 Analysis of Single GEO Satellite LOS Geolocation

From the analysis above we know that the higher the orbit of satellite, the larger the scope of satellite coverage. To enlarge the reconnaissance scope of a satellite, therefore, one method we can adopt is to let the satellite be in a higher orbit. Compared with the LEO (low earth orbit) satellite, the GEO (geostationary orbit) satellite comes with three unique features:

1. Geostationary
2. Far from earth
3. Large visible coverage area for the earth.

If the GEO electronic reconnaissance satellite is set over a hotspot area, $7 / 24$ monitoring on electromagnetic emitters in the area can be realized. It is one of the necessary methods used to carry out reconnaissance and surveillance.
However, when carrying out reconnaissance on GEO, there is a clear difficulty - the great distance between the satellite and the emitter on the earth brings two adverse factors:

1. Deterioration of LOS geolocation accuracy

A geolocation error diagram can be obtained, as shown in Figure 3.18, by using the geolocation solution method mentioned in Section 3.2.2 and the method stated in Section 3.2.3 of calculating the CRLB of the geolocation error.

From Figure 3.18, we can see that as the altitude of the orbit is too high, for a DF error of $1^{\circ}$, the corresponding geolocation accuracy is about 3000 km . As a result, it is almost impossible to decide the exact position of the target emitter - only a vague idea of the area containing the emitter can be determined. The only way to improve the accuracy of the geolocation is to improve the DF measurement accuracy. For example, if we intend to reach geolocation accuracy of 10 km , the DF accuracy should be up to $0.01^{\circ}$. Due to various effects in engineering, such DF accuracy is hard to obtain.
2. Reduction of received signal power

According to the reconnaissance equation of electronic reconnaissance [1], the signal received by the reconnaissance receiver is inversely proportional to the square of the distance $R_{r}^{2}$ between the receiver and the emitter. Comparing this with the LEO electronic
reconnaissance satellite, the signal-to-noise ratio (SNR) lost is

$$
\begin{equation*}
L(\mathrm{~dB})=10 \log \left(\frac{R_{r_{-} \text {high }}^{2}}{R_{r_{-} \text {low }}^{2}}\right), \tag{3.65}
\end{equation*}
$$

where $R_{r_{-} \text {low }}$ is the distance between the LEO satellite and the emitter, while $R_{r_{-} h i g h}$ is the distance between the GEO satellite and the ground emitter. Suppose the height of the LEO satellite is $H=500 \mathrm{~km}$ and the subsatellite points of two satellites are on the equator. Then at the time when the emitter at the subsatellite point is being received, according to expression (3.65), the loss of SNR caused by elevation of the orbit is $L=37 \mathrm{~dB}$. Therefore, the method to increase the antenna aperture can be taken into account to enlarge the antenna gain resulting from the higher SNR of the receiving signals.

### 3.4.2 Geosynchronous Satellite Multitimes LOS Geolocation

What we have discussed in the last section is that far distance leads to bad DF accuracy. A feasible solution is to improve the geolocation accuracy through multiple LOS geolocation of a moving geosynchronous satellite.
For example, for a geosynchronous satellite with inclination of $20^{\circ}$ and eccentricity $e=0.1$, the satellite point performs a ' 8 ' shape motion (period 24 hours) around the mean subsatellite point with no inclination and no eccentricity. This can still meet the demand for continuous reconnaissance coverage, as shown in Figure 3.19.
The geosynchronous electronic reconnaissance satellite makes a good choice. Besides having the merits of both kinds of satellite, the geosynchronous satellite has more advantages:


Figure 3.18 Relation between the GEO LOS geolocation error and the DF error


Figure 3.19 Subsatellite track of a geosynchronous satellite with inclination $20^{\circ}$ and eccentricity $e=0.1$

1. Larger surveillance scope

As the satellite drifts across both south and north of the equator, the coverage area over high latitudes north and south of the equator can be increased.
2. Not restricted by allocation of position on the GEO

There is only one GEO for earth, which holds various GEO satellites needed to be launched by all countries. Positions on the orbit are managed and allocated by the International Telecommunication Union (ITU). Thosehose who need to use the orbit must be approved by the ITU, which is adverse to secrecy of electronic reconnaissance satellites. By using a geosynchronous satellite, this problem can be completely avoided.
3. More measurement points with higher geolocation accuracy

As a geosynchronous satellite keeps moving all the time, the multichannel amplitude comparison DF technique can be used. Signals from the same emitter can be received at different positions of the same reconnaissance beam or at different beams. Using the optimal estimation method, we can obtain higher geolocation accuracy. Therefore, by choosing the geosynchronous orbit, the satellite can locate the position of the emitter at different measurement points, with better geolocation accuracy for the emitter.

Accuracy of single geosynchronous satellite LOS geolocation is not likely to be improved to any large extent. If, for example, an angle measurement accuracy of $0.5^{\circ}$ is assumed in a whole day ( 24 hours) motion process, the theoretical error change of the CRLB of the geolocation for a single time for an emitter in Taipei is as shown in Figure 3.20.
By utilizing the nonstatic features of the geosynchronous satellite and building a multitime statistical method like the average method, the weighted average method, or the EKF method, geolocation accuracy can be improved.
Suppose the satellite intercepts an emitter signal from Taipei every 20 minutes ( 1200 seconds), by adopting the weighted average method, as mentioned in Section 3.3.3, we can make a distribution curve for averaged geolocation points after one day's reconnaissance,


Figure 3.20 Error change curve for CRLB of geolocation in a single time


Figure 3.21 Geolocation distribution after using the weighted average method
as shown in Figure 3.21. After statistical analysis we know that the CEP of geolocation is around 60 km . Therefore, we can conclude that there is a practical value to a certain degree if the geosynchronous satellite is used.

## References

1. Zhou, Y., An, W., Guo, F., et al. (2009) Principles of Electronic Warfare. Beijing: Publishing House of Electronics Industry (in Chinese).
2. Poisel, R.A. (2012) Electronic Warfare Target Location Methods, 2nd edn, Artech House.
3. Hu, L. (2004) Passive Location. Beijing: National Defence Industry Press (in Chinese).
4. Schmidt, R.O. (1986) Multiple emitter location and signal parameter estimation. IEEE Transactions on Antennas and Propagation, 34(3): 276-280.
5. Jiang, C. and Fan, X. (2003) The Introduction of Aerospace Communication and Tracking Technology. Beijing University of Technology Press (in Chinese).
6. Zhou, C., Fan, Y., and Wen, Z. (2003) Military requirements of information operation for GEO electronic reconnaissance satellites. Proceedings of the Special Technology Symposium on Effective Loads of GEO Electronic Reconnaissance Satellites, February 2003 (in Chinese).
7. Liu, H., Liu, Z., Jiang, W., et al. (2009) Fusion algorithm for emitter localization based on satellite carried direction finding system. System Engineering and Electronics, 12(31): 2785-2789 (in Chinese).
8. Gong, W., Feng, D., Xie, K., et al. (2005) Research on the self-calibration approach of satellite carried direction finding localization bias. Aerospace Electronic Warfare, 2(21): 25-30 (in Chinese).
9. Yang, B., Zhang, M., and Li, L. (2009) Algorithm for single-satellite DOA localization based on model WGS284. Aerospace Electronic Warfare, 25: 24-26 (in Chinese).
10. Yuan, X. (2005) Research on ground target direction finding localization by remote-sensing satellite. Space Electronic Technology, 1-9 (in Chinese).
11. Kay, S. (1998) Fundamentals of Statistical Signal Processing, Volume I: Estimation Theory/Volume II: Detection Theory, Prentice Hall PTR.
12. Bar-Shalom, Y., Li, R. X., and Kjrubarajan, T. (2001) Estimation with Applications to Tracking and Navigation. New York: John Wiley \& Sons, Inc..

## 4

## Multiple Satellites Geolocation Based on TDOA Measurement

In three-dimensional space, according to the signal time difference of arrival (TDOA) of a transmitter on the earth between two geometric separated satellites, we can draw a revolution hyperboloid taking the two satellites as the focus. For 3D geolocation, at least four observers are required to simultaneously receive emitter signals to obtain three groups of uncorrelated TDOA, according to which three hyperboloids indicating location of the emitter can be formed and the intersection point is the location of the emitter [1, 2].
To reduce the number of required observers, prior knowledge about the location of the emitter on the earth's surface can be utilized. In this way, the location of the ground emitter can be achieved by obtaining two TDOAs with only three satellites. Two revolution hyperboloids can be formed through the two TDOAs. By intersecting the two hyperboloids, a location curve that intersects the earth's surface can be formed to obtain two intersection points. After the intersection point (ambiguity point) located on the other side of the earth is deleted, the location of the emitter can be determined [3-9].
Prior knowledge of the earth's surface is required during the above location process, so it is very important to describe the earth model accurately. In general, the WGS-84 ellipse earth model of the earth can be employed to describe the earth's surface. This chapter firstly sets out the most simple location algorithm of the regular spherical model and then analyzes the multiple satellite geolocation method based on the WGS-84 earth surface model to research problems of ambiguity and a no-solution situation of geolocation. Last, the error of three-satellite geolocation is analyzed and several calibration methods using multiple ground stations to calibrate the errors of the satellite TDOA system and satellite position are proposed.

[^0]
### 4.1 Three-Satellite Geolocation Based on a Regular Sphere

Compared with the complicated average earth surface model defined by the WGS-84 coordinate system, the earth surface model of a regular sphere is simpler [10]. This section firstly introduces the three-satellite TDOA algorithm based on a regular sphere and then introduces the multisatellite TDOA algorithm, and last the osculation error of the regular spherical model is analyzed.

### 4.1.1 Three-Satellite Geolocation Solution Method

Assume in ECEF (earth-center earth-fixed) coordinates, the geocentric Cartesian coordinates of three satellites and the emitter at a certain moment respectively are $O_{0}\left(x_{0}, y_{0}, z_{0}\right)$, $O_{1}\left(x_{1}, y_{1}, z_{1}\right), O_{2}\left(x_{2}, y_{2}, z_{2}\right)$, and $T(x, y, z)$, as shown in Figure 4.1.
By comparing the TOA (time of arrival) of the received signal at three satellites, two TDOAs can be obtained:

$$
\begin{equation*}
T D O A_{i}=T O A_{i}-T O A_{0} \triangleq \Delta t_{i} \quad(i=1,2) . \tag{4.1}
\end{equation*}
$$

The defined range is

$$
\begin{equation*}
r_{i}=\sqrt{\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}+\left(z-z_{i}\right)^{2}} \quad(i=0,1,2) \tag{4.2}
\end{equation*}
$$



Figure 4.1 Schematic diagram of the three-satellite geolocation

Essentially, the TDOAs measured are the range differences [8]:

$$
\left.\begin{array}{l}
r_{1}-r_{0}=\Delta r_{1}=c \Delta t_{1}  \tag{4.3}\\
r_{2}-r_{0}=\Delta r_{2}=c \Delta t_{2}
\end{array}\right\}
$$

where $c$ is the propagating speed of the electromagnetic wave.
Substitute the range definition expression into the TDOA expressions (4.3). If it is then rearranged and squared, one can obtain

$$
\begin{equation*}
\left(x_{0}-x_{i}\right) x+\left(y_{0}-y_{i}\right) y+\left(z_{0}-z_{i}\right) z=k_{i}+r_{0} \Delta r_{i} \quad(i=1,2), \tag{4.4}
\end{equation*}
$$

where

$$
k_{i}=\frac{1}{2}\left[\left(x_{0}^{2}+y_{0}^{2}+z_{0}^{2}\right)-\left(x_{i}^{2}+y_{i}^{2}+z_{i}^{2}\right)+\Delta r_{i}^{2}\right]
$$

Suppose the earth is a regular sphere with a radius of $R$. Then the earth's surface conforms to the expression

$$
\begin{equation*}
x^{2}+y^{2}+z^{2}=R^{2} . \tag{4.5}
\end{equation*}
$$

After subtracting the expression (4.5) from the first $r_{0}^{2}$ squared expression of Equation (4.2) when $i=0$, we can obtain

$$
\begin{equation*}
x_{0} x+y_{0} y+z_{0} z=k_{3}-\frac{1}{2} r_{0}^{2} \tag{4.6}
\end{equation*}
$$

where

$$
k_{3}=\frac{1}{2}\left(x_{0}^{2}+y_{0}^{2}+z_{0}^{2}+R^{2}\right) .
$$

Combine expressions (4.4) and (4.6) as follows:

$$
\begin{equation*}
\mathbf{A X}=\mathbf{F} \tag{4.7}
\end{equation*}
$$

where

$$
\begin{aligned}
\mathbf{A} & =\left[\begin{array}{ccc}
x_{0}-x_{1} & y_{0}-y_{1} & z_{0}-z_{1} \\
x_{0}-x_{2} & y_{0}-y_{2} & z_{0}-z_{2} \\
x_{0} & y_{0} & z_{0}
\end{array}\right], \\
\mathbf{x} & =[x, y, z]^{\mathrm{T}}, \\
\mathbf{F} & =\left[\begin{array}{c}
k_{1}+r_{0} \Delta r_{1} \\
k_{2}+r_{0} \Delta r_{2} \\
k_{3}-\frac{1}{2} r_{0}^{2}
\end{array}\right] .
\end{aligned}
$$

When $\mathbf{A}$ is invertible, the following expression can be obtained:

$$
\begin{equation*}
\widehat{\mathbf{X}}=\mathbf{A}^{-1} \mathbf{F} \triangleq\left[a_{i j}\right]_{3 \times 3} \mathbf{F} \tag{4.8}
\end{equation*}
$$

Let

$$
\widehat{\mathbf{X}}=\left[\begin{array}{l}
\hat{x}  \tag{4.9}\\
\hat{y} \\
\hat{z}
\end{array}\right]=\left[\begin{array}{l}
m_{1}+n_{1} r_{0}+p_{1} r_{0}{ }^{2} \\
m_{2}+n_{2} r_{0}+p_{2} r_{0}^{2} \\
m_{3}+n_{3} r_{0}+p_{3} r_{0}^{2}
\end{array}\right],
$$

where

$$
\left.\begin{array}{l}
m_{1}=a_{11} k_{1}+a_{12} k_{2}+a_{13} k_{3}  \tag{4.10}\\
n_{1}=a_{11} \Delta r_{1}+a_{12} \Delta r_{2} \\
p_{1}=-\frac{1}{2} a_{13} \\
m_{2}=a_{21} k_{1}+a_{22} k_{2}+a_{23} k_{3} \\
n_{2}=a_{21} \Delta r_{1}+a_{22} \Delta r_{2} \\
p_{2}=-\frac{1}{2} a_{23} \\
m_{3}=a_{31} k_{1}+a_{32} k_{2}+a_{33} k_{3} \\
n_{3}=a_{31} \Delta r_{1}+a_{32} \Delta r_{2} \\
p_{3}=-\frac{1}{2} a_{33}
\end{array}\right\} .
$$

Substitute Equation (4.9) into Equation (4.5) to obtain a quartic equation with $r_{0}$ unknown:

$$
\begin{equation*}
s_{1} r_{0}^{4}+s_{2} r_{0}^{3}+s_{3} r_{0}^{2}+s_{4} r_{0}+s_{5}=0 \tag{4.11}
\end{equation*}
$$

where all factors are as follows:

$$
\left.\begin{array}{l}
s_{1}=p_{1}^{2}+p_{2}^{2}+p_{3}^{2} \\
s_{2}=2 n_{1} p_{1}+2 n_{2} p_{2}+2 n_{3} p_{3} \\
s_{3}=n_{1}^{2}+n_{2}^{2}+n_{3}^{2}-1+2\left(m_{1}-x_{0}\right) p_{1}+2\left(m_{2}-y_{0}\right) p_{2}+2\left(m_{3}-z_{0}\right) p_{3} \\
s_{4}=2\left(m_{1}-x_{0}\right) n_{1}+2\left(m_{2}-y_{0}\right) n_{2}+2\left(m_{3}-z_{0}\right) n_{3} \\
s_{5}=\left(m_{1}-x_{0}\right)^{2}+\left(m_{2}-y_{0}\right)^{2}+\left(m_{3}-z_{0}\right)^{2}
\end{array}\right\} .
$$

Here $\widehat{r}_{0}$ is solved based on expression (4.11); substitute expression (4.9) to obtain the geocentric Cartesian coordinates ( $x, y, z$ ) of the emitter under the spherical model assumption. Generally speaking, the intersection line of two TDOA curves intersects with the spherical surface at two points, so there is location ambiguity. The problem of ambiguity will be analyzed in Section 4.3.

### 4.1.2 Multisatellite TDOA Geolocation Method

A multisatellite TDOA geolocation method was proposed by Ho and Chan in 1997. For such a method, a three-satellite geolocation is only one of special cases in their paper [7]. Suppose that the location of an emitter on the earth is expressed as $\mathbf{u}=[x, y, z]^{\mathrm{T}}$ in the geocentric coordinates system and there are $M(M \geq 3)$ measurement satellites, whose positions are $\mathbf{s}_{i}=\left[x_{i}, y_{i}, z_{i}\right]^{\mathrm{T}}$; thus $M_{1}$ TDOAs can be obtained: $d_{i, 1}(i=2,3, \ldots, M)$.
Suppose that $r_{i}$ represents the range between the $i$ th satellite and the emitter:

$$
\begin{equation*}
r_{i}=\left|\mathbf{s}_{i}-\mathbf{u}\right|=\sqrt{\left(x_{i}-x\right)^{2}+\left(y_{i}-y\right)^{2}+\left(z_{i}-z\right)^{2}} \quad(i=1,2, \ldots, M) . \tag{4.12}
\end{equation*}
$$

If $c$ represents the signal propagation speed and $c d_{i, 1}$ is the range difference, the multisatellite TDOA expression can be expressed as follows:

$$
\begin{equation*}
r_{i, 1}=c d_{i, 1}=r_{i}-r_{1} \quad(i=2,3, \ldots, M) \tag{4.13}
\end{equation*}
$$

Suppose the sum of the altitude of the emitter and the radius of the earth is $r$; the location of the emitter satisfies:

$$
\begin{equation*}
\mathbf{u}^{T} \mathbf{u}=r^{2} \tag{4.14}
\end{equation*}
$$

In fact, the measured value is always corrupted by noise, which leads to a geolocation error. For the purpose of location accuracy analysis, suppose that $\mathbf{d}=\left[d_{2,1}, d_{3,1}, \ldots, d_{M, 1}\right]^{\mathrm{T}}$ is the measurement vector of TDOA, the measurement model is $\mathbf{d}=\mathbf{d}^{0}+\Delta \mathbf{d}, E[\Delta \mathbf{d}]=0, E\left[\Delta \mathbf{d} \Delta \mathbf{d}^{\mathrm{T}}\right]=$ $\mathbf{Q}_{t}$, and $\mathbf{d}^{0}$ expresses the true value of the TDOA vector.
Square expression (4.13) and then substitute expression (4.12), which yields

$$
\begin{equation*}
r_{i, 1}^{2}+2 r_{i, 1} r_{1}+r_{1}^{2}=r^{2}+\mathbf{s}_{i}^{\mathrm{T}} \mathbf{s}_{i}-2 \mathbf{s}_{i}^{\mathrm{T}} \mathbf{u}(i=2,3, \ldots, M) \tag{4.15}
\end{equation*}
$$

From expression (4.12),

$$
\begin{equation*}
r_{1}^{2}=r^{2}+\mathbf{s}_{1}^{\mathrm{T}} \mathbf{s}_{1}-2 \mathbf{s}_{1}^{\mathrm{T}} \mathbf{u} . \tag{4.16}
\end{equation*}
$$

Substitute expression (4.16) into expression (4.15) yields

$$
\begin{equation*}
r_{i, 1}^{2}+2 r_{i, 1} r_{1}=\mathbf{s}_{i}^{\mathrm{T}} \mathbf{s}_{i}-\mathbf{s}_{1}^{\mathrm{T}} \mathbf{s}_{1}-2\left(\mathbf{s}_{i}-\mathbf{s}_{1}\right)^{\mathrm{T}} \mathbf{u}(i=2,3, \ldots, M) \tag{4.17}
\end{equation*}
$$

It can be found that the root of Equations (4.13) and (4.14) is the same as those of expressions (4.16), (4.17), and (4.14). To solve location $\mathbf{u}$, express $\mathbf{u}$ as $r_{1}$ via expressions (4.16) and (4.17), and obtain $r_{1}$ through expression (4.14), thereby solving the location $\mathbf{u}$.
For the number of satellites, there are two cases [7]:

1. When the number of satellites $M=3$, the location solution and its covariance are as follows. Rewrite expressions (4.16) and (4.17) into matrix form to obtain the solution of $\mathbf{u}$ expressed using $r_{1}$ :

$$
\begin{equation*}
\mathbf{u}=\mathbf{G}_{1}^{-1} \mathbf{h} \tag{4.18}
\end{equation*}
$$

where

$$
\mathbf{G}_{1}=-2\left[\begin{array}{c}
\mathbf{s}_{1}^{\mathrm{T}} \\
\mathbf{s}_{2}^{\mathrm{T}}-\mathbf{s}_{1}^{\mathrm{T}} \\
\mathbf{s}_{3}^{\mathrm{T}}-\mathbf{s}_{1}^{\mathrm{T}}
\end{array}\right], \mathbf{h}=\left[\begin{array}{lll}
-r^{2}-\mathbf{s}_{1}^{\mathrm{T}} \mathbf{s}_{1} & 0 & 1 \\
r_{2,1}^{2}-\mathbf{s}_{2}^{\mathrm{T}} \mathbf{s}_{2}+\mathbf{s}_{1}^{\mathrm{T}} \mathbf{s}_{1} & 2 r_{2,1} & 0 \\
r_{3,1}^{2}-\mathbf{s}_{3}^{\mathrm{T}} \mathbf{s}_{3}+\mathbf{s}_{1}^{\mathrm{T}} \mathbf{s}_{1} & 2 r_{3,1} & 0
\end{array}\right]\left[\begin{array}{c}
1 \\
r_{1} \\
r_{1}^{2}
\end{array}\right] .
$$

Substitute Equation (4.18) into Equation (4.14) to obtain a quartic polynomial expression. Solve the expression and substitute $r_{1}$ into Equation (4.18), thereby solving the estimated location $\mathbf{u}$ of the emitter. The polynomial normally has more than one positive root; in this case, use expression (4.13) to eliminate uncertainty - sometimes prior knowledge about the emitter is required for this purpose.

During solution, the matrix $\mathbf{G}_{1}$ is required to be not invertible, which is equivalent to the fact that any three points among $\left\{\mathbf{O}, \mathbf{s}_{1}, \mathbf{s}_{2}, \mathbf{s}_{3}\right\}$ cannot be in a straight line, where $\mathbf{O}$ represents the earth's center. In addition, the spatial distance between three satellites should be sufficient to eliminate an ill-conditioned matrix.

The location solution is obtained from expressions (4.13) and (4.14). Take $\mathbf{u}$ as a variable and differentiate the location solution of Equation 4.18; then calculate it at the real value, which yields

$$
\mathbf{H} \Delta \mathbf{u}=c\left[\begin{array}{c}
\Delta \mathbf{d}  \tag{4.19}\\
0
\end{array}\right],
$$

where

$$
\mathbf{H}=-\left[\begin{array}{c}
\left(\mathbf{s}_{2}-\mathbf{u}^{0}\right)^{\mathrm{T}} / r_{2}^{0}-\left(\mathbf{s}_{1}-\mathbf{u}^{0}\right)^{\mathrm{T}} / r_{1}^{0} \\
\left(\mathbf{s}_{3}-\mathbf{u}^{0}\right)^{\mathrm{T}} / r_{3}^{0}-\left(\mathbf{s}_{1}-\mathbf{u}^{0}\right)^{\mathrm{T}} / r_{1}^{0} \\
\mathbf{u}^{0 \mathrm{~T}}
\end{array}\right] .
$$

When the number of satellites is $M=3$, the covariance matrix of the geolocation solution error is

$$
\boldsymbol{\Psi}=\operatorname{cov}(\mathbf{u})=c^{2} \mathbf{H}^{-1}\left[\begin{array}{ll}
\mathbf{Q}_{t} & \mathbf{0}  \tag{4.20}\\
\mathbf{0}^{\mathrm{T}} & 0
\end{array}\right] \mathbf{H}^{-\mathrm{T}} .
$$

2. When the number of satellites $M>3$, the geolocation methods are as follows. This case is a redundant case, that is, the number of expressions is more than the number of unknowns. The solution can be obtained by minimizing the error of expression (4.16), that is, the least squares estimation or the maximum likelihood estimation under the two nonlinear constraints of Equations (4.16) and (4.14). The cost function is as follows:

$$
\begin{align*}
\xi \equiv & \left(\mathbf{h}-\mathbf{G}_{1} \mathbf{u}-\mathbf{g}_{2} \mathbf{r}_{1}\right)^{\mathrm{T}} \mathbf{W}\left(\mathbf{h}-\mathbf{G}_{1} \mathbf{u}-\mathbf{g}_{2} \mathbf{r}_{1}\right) \\
& +\lambda_{1}\left(2 \mathbf{s}_{1}^{\mathrm{T}} \mathbf{u}-\mathbf{s}_{1}^{\mathrm{T}} \mathbf{s}_{1}-r^{2}+r_{1}^{2}\right)+\lambda_{2}\left(\mathbf{u}^{\mathrm{T}} \mathbf{u}-r^{2}\right), \tag{4.21}
\end{align*}
$$

where
and $\mathbf{W}$ is the weighting matrix. By differentiating expression (4.21) with respect to $\mathbf{u}$ and $r_{1}$ and letting them equal 0 , one can obtain

$$
\begin{equation*}
\mathbf{u}=\mathbf{G}_{4}\left(\mathbf{G}_{1}^{\mathrm{T}} \mathbf{W} \mathbf{G}_{5} \mathbf{r}_{1}-\lambda_{1} \mathbf{s}_{1}\right), \tag{4.22}
\end{equation*}
$$

where

$$
\begin{align*}
& \mathbf{G}_{4}=\left(\mathbf{G}_{1}^{\mathrm{T}} \mathbf{W} \mathbf{G}_{1}+\lambda_{2} \mathbf{I}\right)^{-1}, \mathbf{r}_{1}=\left[1, r_{1}, r_{1}^{2}\right]^{\mathrm{T}}, \mathbf{G}_{5}=\left[\mathbf{h},-\mathbf{g}_{2}, \mathbf{0}\right], \text { and } \\
& -\mathbf{g}_{2}^{\mathrm{T}} \mathbf{W}\left(\mathbf{G}_{5} \mathbf{r}_{1}-\mathbf{G}_{1} \mathbf{u}\right)+\lambda_{1} r_{1}=0 . \tag{4.23}
\end{align*}
$$

Let $\mathbf{g}_{3}=\left[\mathbf{s}_{1}^{\mathrm{T}} \mathbf{s}_{1}+r^{2}, 0,-1\right]^{\mathrm{T}}$. The expression (4.16) can be transformed to $2 \mathbf{s}_{1}^{\mathrm{T}} \mathbf{u}=\mathbf{g}_{3}^{\mathrm{T}} \mathbf{r}_{1}$. Substitute $\mathbf{u}$ in expression (4.22) into expression (4.16), which yields

$$
\begin{equation*}
\lambda_{1}=\mathbf{g}_{6}^{\mathrm{T}} \mathbf{r}_{1} \text { and } \quad \mathbf{g}_{6}^{\mathrm{T}}=\frac{2 \mathbf{s}_{1}^{\mathrm{T}} \mathbf{G}_{4} \mathbf{G}_{1}^{\mathrm{T}} \mathbf{W} \mathbf{G}_{5}-\mathbf{g}_{3}^{\mathrm{T}}}{2 \mathbf{s}_{1}^{\mathrm{T}} \mathbf{G}_{4} \mathbf{s}_{1}} \tag{4.24}
\end{equation*}
$$

Substitute expression (4.24) into expression (4.22), which yields

$$
\begin{equation*}
\mathbf{u}=\mathbf{G}_{7} \mathbf{r}_{1} \text { and } \mathbf{G}_{7}=\mathbf{G}_{4}\left(\mathbf{G}_{1}^{\mathrm{T}} \mathbf{W} \mathbf{G}_{5}-\mathbf{s}_{1} \mathbf{g}_{6}^{\mathrm{T}}\right) . \tag{4.25}
\end{equation*}
$$

Substitute expressions (4.24) and (4.25) into expression (4.23) to obtain a polynomial expression of $r_{1}$ :

$$
\begin{equation*}
r_{1} \mathbf{g}_{6}^{\mathrm{T}} \mathbf{r}_{1}-\mathbf{g}_{8}^{\mathrm{T}} \mathbf{r}_{1}=0 \tag{4.26}
\end{equation*}
$$

where $\mathbf{g}_{8}^{\mathrm{T}}=\mathbf{g}_{2}^{\mathrm{T}} \mathbf{W}\left(\mathbf{G}_{5}-\mathbf{G}_{1} \mathbf{G}_{7}\right)$. When $\lambda_{2}$ is specified, use the above expression to solve $r_{1}$. In most circumstances, only one positive root can be solved. Substitute the positive $r_{1}$ into expression (4.25) to obtain the estimated location of the emitter with $\lambda_{2}$ as its parameter. Use Equation (4.14) to solve $\lambda_{2}$. If expression (4.26) has more than one positive root, prior information about the emitter is required to select the correct roots.

The Newton method is an efficient one to solve the expression $\rho\left(\lambda_{2}\right)=\mathbf{u}^{\mathrm{T}} \mathbf{u}-r^{2}$. Since nonlinearity of such an expression leads to more solutions, when $\lambda_{2}=0$ the altitude information is not used. Therefore, if the Newton method is used, the initial value of $\lambda_{2}$ is 0 . According to $\mathbf{G}_{4}$ in expression (4.22), it is shown that $\lambda_{2}$ is only used to modify the eigenvalue of such a matrix, so matrix inversion can be avoided. Therefore the Newton method is more effective than the grid search method and other iteration methods requiring an initial value (such as the Taylor expanding method).

### 4.1.3 CRLB of a Multisatellite TDOA Geolocation Error

To compute the Cramér-Rao lower bound (CRLB) under constraint conditions, reference [7] gives the estimation $\mathbf{u}$ for an unbiased constraint as

$$
\begin{equation*}
\operatorname{cov}(\mathbf{u})_{\min }=\mathbf{J}^{-1}-\left.\mathbf{J}^{-1} \mathbf{F}\left(\mathbf{F}^{\mathrm{T}} \mathbf{J}^{-1} \mathbf{F}\right)^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{J}^{-1}\right|_{\mathbf{u}=\mathbf{u}^{0}} \tag{4.27}
\end{equation*}
$$

where $\mathbf{J}$ is the Fisher information matrix and $\mathbf{F}$ is the gradient matrix of unknown variable constraint expressions. The CRLB without constraint conditions is $\mathbf{J}^{-1}$. It is shown that from expression (4.27) of prior information the constraint condition can decrease the CRLB. In this section, accuracy of the foregoing solution is compared with the CRLB under constraint conditions.
When the measurement noise is Gaussian noise, the Fisher information matrix is

$$
\begin{equation*}
\mathbf{J}_{T D O A}=\left.\left(\frac{\partial \mathbf{d}^{0 \mathrm{~T}}}{\partial \mathbf{u}} \mathbf{Q}_{t}^{-1} \frac{\partial \mathbf{d}^{0}}{\partial \mathbf{u}}\right)\right|_{\mathbf{u}=\mathbf{u}^{0}} \tag{4.28}
\end{equation*}
$$

where

$$
\frac{\partial \mathbf{d}^{0}}{\partial \mathbf{u}}=-\left[\begin{array}{c}
\left(\mathbf{s}_{2}-\mathbf{u}\right)^{\mathrm{T}} / r_{2}-\left(\mathbf{s}_{1}-\mathbf{u}\right)^{\mathrm{T}} / r_{1} \\
\left(\mathbf{s}_{3}-\mathbf{u}\right)^{\mathrm{T}} / r_{3}-\left(\mathbf{s}_{1}-\mathbf{u}\right)^{\mathrm{T}} / r_{1} \\
\vdots \\
\left(\mathbf{s}_{M}-\mathbf{u}\right)^{\mathrm{T}} / r_{M}-\left(\mathbf{s}_{1}-\mathbf{u}\right)^{\mathrm{T}} / r_{1}
\end{array}\right] .
$$

With respect to the location with earth sphere constraint conditions, it can be derived from expression (4.14) that $\mathbf{F}$ is equal to $\mathbf{u}$. To find the CRLB of the constraint condition, substitute Equation (4.28) into Equation (4.27).
Suppose that the location of the emitter is $\left(75.9^{\circ} \mathrm{W}, 45.35^{\circ} \mathrm{N}\right)$, the altitude of the emitter is zero, and the local radius of the earth is $r=6367.287 \mathrm{~km}$. The receiver is a geostationary satellite 42164 km away from the earth's center, with the location being $s_{1}=\left(53.0^{\circ} \mathrm{W}, 2.0^{\circ} \mathrm{N}\right)$, $s_{2}=\left(47.0^{\circ} \mathrm{W}, 0.0^{\circ} \mathrm{N}\right)$, and $s_{3}=\left(53.0^{\circ} \mathrm{W}, 0.0^{\circ} \mathrm{N}\right)$, respectively. The measured TDOA error is Gaussian noise and the covariance matrix of Gaussian noise is $Q_{t}$, whose diagonal element is $c^{2} \sigma_{d}^{2}$. The other elements are $0.5 c^{2} \sigma_{d}^{2}$, where $\sigma_{d}^{2}$ is the variance of the TDOA measurement error. The relationship between the theoretical and actual geolocation error if $M=3$ (the


Figure 4.2 Relationship between the error of three HEO satellite geolocations and the TDOA measurement error
methods in Section 4.1.2 are used) can be obtained with RMS (root mean square) of the TDOA measurement error, as shown in Figure 4.2.
From Figure 4.2, it is shown that the geolocation error is increased with an increasing TDOA error. When $M=3$, the algorithm referred to in Section 4.1.2 can achieve the CRLB.

### 4.1.4 Osculation Error of the Spherical Earth Model

In this section, osculation bias of a regular spherical earth model and the resulting geolocation bias are analyzed.
Take the WGS-84 earth ellipsoid as the reference to investigate the osculation bias of a regular sphere (see Figure 4.3). For the earth ellipsoid, a spheroid with its radius having a geocentric radius vector of point P on the earth's surface is used. $\mathrm{E}^{\prime}$ is a point different from point P with a normal direction of a prime vertical circle $\mathrm{O}^{\prime} \mathrm{E}^{\prime}$, which intersects the spherical model at point D ; the line OD between the earth's center and point D intersects the earth ellipsoid at point E . It can be concluded that the osculation error of a regular sphere can be measured by the altitude difference $\mathrm{DE}^{\prime}$ [8].
Since the angle bias between the local normal direction of the prime vertical circle $\mathrm{O}^{\prime} \mathrm{E}^{\prime}$ and the geocentric direction OE of point D is no more than 4 mrad , the length difference will be no more than $4 \%$ of $O^{\prime} E^{\prime}$. Therefore, the direction difference may be omitted when calculating the altitude. Suppose the radius of the regular sphere is $R$. The altitude $\mathrm{DE}^{\prime}$ can be approximately


Figure 4.3 Osculation bias of a regular sphere
expressed as

$$
\begin{equation*}
\mathrm{DE}^{\prime} \approx \mathrm{DE}=\mathrm{OE}-R \tag{4.29}
\end{equation*}
$$

To minimize the osculation bias, the radius of the spherical model will be close to the geocentric radius vector at a local emitter. Due to the constraint by height of the satellite, the observation beam angle, and the minimum observation angle, the location where three satellites intercepts the emitter will be adjacent to the subsatellite point; therefore, the radius vector of the subsatellite point can be taken as the radius of the spherical model to achieve the optimum spherical approximation.
After converting the osculation bias of the spherical model into the altitude error, the altitude error analytic method can be used to investigate the influence caused by the spherical osculation error. In Figure 4.4, the osculation altitude error of the spherical model and the geolocation error distribution incurred therefrom are provided, where the simulation parameters are: satellite height, 1000 km ; subsatellite locations of the satellites respectively are $\mathrm{O}_{0}$ $\left(125.21^{\circ} \mathrm{E}, 24.44^{\circ} \mathrm{N}\right), \mathrm{O}_{1}\left(125.68^{\circ} \mathrm{E}, 25.09^{\circ} \mathrm{N}\right)$, and $\mathrm{O}_{2}\left(125.99^{\circ} \mathrm{E}, 24.44^{\circ} \mathrm{N}\right)$. The radius of the spherical model is the radius of the subsatellite point of the primary satellite $\mathrm{O}_{0}$. In Figure 4.4b, '* represents the location of the subsatellite points and the marks on the curves represent the location accuracy in meters.
It is revealed from the simulation results that there is about $3^{\circ}(300 \mathrm{~km})$ of deviation on latitude, the altitude error will be up to 1 km , and the geolocation error caused thereby will amount to $500-600 \mathrm{~m}$, which is similar to the case with the random measurement geolocation error with typical measurement accuracy. It can be concluded that the osculation bias of the spherical model is relatively obvious. Therefore, if a highly accurate location is required, the simple regular spherical model cannot be used and the more accurate WGS-84 earth ellipsoid method should be employed.


Figure 4.4 Osculation altitude difference and bias distribution of a spherical model. (a) Relation of height difference (meter) between a regular sphere and an ellipsoid and latitude and (b) osculation error using a regular spherical model in a three-satellite TDOA geolocation

### 4.2 Three-Satellite Geolocation Based on the WGS-84 Earth Surface Model

The geoidal surface of the earth is an irregular sphere that osculates with an approximate ellipsoid with a long equatorial radius and a short polar radius. It is therefore obvious that errors will be caused if the sphere is taken as the regular sphere surface model and so the more accurate ellipse model of the earth should be employed for accurate location. For this purpose, the ellipse model of the earth's sphere defined by the WGS-84 earth coordinates system is used.

From the earth ellipsoid defined by expression (2.30) we get

$$
\begin{equation*}
\frac{x^{2}}{(N+H)^{2}}+\frac{y^{2}}{(N+H)^{2}}+\frac{z^{2}}{\left[N\left(1-e^{2}\right)+H\right]^{2}}=1 \tag{4.30}
\end{equation*}
$$

where $N$ is the radius of curvature in the prime vertical of the point where the emitter is located. Expression (4.30) defines a group of ellipsoid surfaces with altitude as the parameter. If the altitude $H$ is unknown, it is impossible to get the emitter location from the TDOA expression (4.3) and ellipsoid expression (4.30). Therefore, the location will not be implemented until the altitude parameter is able to determine the geolocation ellipsoid surface. To achieve this, prior knowledge about the emitter is required, that is, which location ellipsoid surface will be located. In general, for the emitter located adjacent to the geoidal surface (such as marine emitters), $H=0$ can easily be determined. Error is introduced into the prior assumption about the emitter altitude; such errors will be analyzed in Section 4.4.2.
Assume that the altitude is zero. The location ellipsoid surface (Equation (4.30)) can be expressed in standard ellipsoid form as

$$
\begin{equation*}
\frac{x^{2}}{a^{2}}+\frac{y^{2}}{a^{2}}+\frac{z^{2}}{\left(1-e^{2}\right) a^{2}}=1 \tag{4.31}
\end{equation*}
$$

where $a$ is the major axis of the earth. Combine it and the TDOA expression (4.3) to use three-satellite geolocation expressions with a zero-altitude ellipsoid surface as the geolocation plane:

$$
\left.\begin{array}{l}
\sqrt{\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}+\left(z-z_{1}\right)^{2}}-\sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2}}=c \Delta t_{1}  \tag{4.32}\\
\sqrt{\left(x-x_{2}\right)^{2}+\left(y-y_{2}\right)^{2}+\left(z-z_{2}\right)^{2}}-\sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2}}=c \Delta t_{2} \\
x^{2}+y^{2}+z^{2} /\left(1-e^{2}\right)=a^{2}
\end{array}\right\} .
$$

Thenalytical method and the iteration method may be used to solve such expressions, which are respectively introduced below.

### 4.2.1 Analytical Method

Reference [7] provides an analytical method which is introduced below. For the location expression (4.32), consider that

$$
\begin{equation*}
r_{0}^{2}=\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2} . \tag{4.33}
\end{equation*}
$$

The first and second formulas in expression (4.32) can be translated into

$$
\begin{equation*}
\left(x_{0}-x_{i}\right) x+\left(y_{0}-y_{i}\right) y+\left(z_{0}-z_{i}\right) z=k_{i}+r_{0} \Delta r_{i} \quad(i=1,2) \tag{4.34}
\end{equation*}
$$

Through the expression above, a linear combination of $z$ and $r_{0}$ can be used to express $x, y$, that is,

$$
\left.\begin{array}{l}
x=a_{1} r_{0}+b_{1} z+c_{1}  \tag{4.35}\\
y=a_{2} r_{0}+b_{2} z+c_{2}
\end{array}\right\}
$$

where $a_{1}, b_{1}, c_{1}, a_{2}, b_{2}, c_{2}$ can be expressed with known constants as follows:

$$
\left.\begin{array}{l}
a_{1}=\left[\left(y_{0}-y_{2}\right) \Delta r_{1}-\left(y_{0}-y_{1}\right) \Delta r_{2}\right] / J  \tag{4.36}\\
b_{1}=\left[\left(y_{0}-y_{1}\right)\left(z_{0}-z_{2}\right)-\left(y_{0}-y_{2}\right)\left(z_{0}-z_{1}\right)\right] / J \\
c_{1}=\left[k_{1}\left(y_{0}-y_{2}\right)-k_{2}\left(y_{0}-y_{1}\right)\right] / J \\
a_{2}=\left[\left(x_{0}-x_{1}\right) \Delta r_{2}-\left(x_{0}-x_{2}\right) \Delta r_{1}\right] / J \\
b_{2}=\left[\left(x_{0}-x_{2}\right)\left(z_{0}-z_{1}\right)-\left(x_{0}-x_{1}\right)\left(z_{0}-z_{2}\right)\right] / J \\
c_{2}=\left[k_{2}\left(x_{0}-x_{1}\right)-k_{1}\left(x_{0}-x_{2}\right)\right] / J
\end{array}\right\},
$$

where

$$
J=\left(x_{0}-x_{1}\right)\left(y_{0}-y_{2}\right)-\left(x_{0}-x_{2}\right)\left(y_{0}-y_{1}\right)
$$

The third formula in expressions (4.32) and (4.33) can be used to define $E^{\prime}=e^{2} /\left(1-e^{2}\right)$ to get

$$
\begin{equation*}
E^{\prime} z^{2}+2 x_{0} x+2 y_{0} y+2 z_{0} z=a^{2}-r_{0}^{2}+x_{0}^{2}+y_{0}^{2}+z_{0}^{2} . \tag{4.37}
\end{equation*}
$$

Substituting expression (4.35) and simplifying yields

$$
\begin{gather*}
E^{\prime} z^{2}+\left(2 x_{0} b_{1}+2 y_{0} b_{2}+2 z_{0}\right) z+\left(2 x_{0} a_{1}+2 y_{0} a_{2}\right) r_{0}+2 x_{0} c_{1}+2 y_{0} c_{2} \\
-a^{2}+r_{0}^{2}-x_{0}^{2}-y_{0}^{2}-z_{0}^{2}=0 . \tag{4.38}
\end{gather*}
$$

Replace the known constant to obtain

$$
\begin{equation*}
z^{2}+m z+n r_{0}+p r_{0}^{2}+q=0 \tag{4.39}
\end{equation*}
$$

where

$$
\left.\begin{array}{l}
m=\frac{2 x_{0} b_{1}+2 y_{0} b_{2}+2 z_{0}}{E^{\prime}} \\
n=\frac{2 x_{0} a_{1}+2 y_{0} a_{2}}{E^{\prime}} \\
p=\frac{1}{E^{\prime}} \\
q=\frac{2 x_{0} c_{1}+2 y_{0} c_{2}-a^{2}-x_{0}^{2}-y_{0}^{2}-z_{0}^{2}}{E^{\prime}}
\end{array}\right\}
$$

Solving the quadratic expressions of $z$ in expression (4.39) yields

$$
\begin{equation*}
z=\frac{-m \pm \sqrt{m^{2}-4\left(n r_{0}+p r_{0}^{2}+q\right)}}{2} . \tag{4.40}
\end{equation*}
$$

Let

$$
\left.\begin{array}{l}
A=-\frac{m}{2} \\
B=\frac{m^{2}}{4}-q \\
C=-n \\
D=-p
\end{array}\right\}
$$

We can obtain

$$
\begin{equation*}
z=A \pm \sqrt{B+C r_{0}+D r_{0}^{2}} \tag{4.41}
\end{equation*}
$$

Substitute expression (4.41) inversely into expression (4.35) to yield

$$
\left.\begin{array}{l}
x=a_{1} r_{0}+b_{1}\left(A \pm \sqrt{B+C r_{0}+D r_{0}^{2}}\right)+c_{1}  \tag{4.42}\\
y=a_{2} r_{0}+b_{2}\left(A \pm \sqrt{B+C r_{0}+D r_{0}^{2}}\right)+c_{2} \\
z=A \pm \sqrt{B+C r_{0}+D r_{0}^{2}}
\end{array}\right\} .
$$

According to expression (4.42), $x, y$, and $z$ can be expressed by functions of $r_{0}$. Therefore, if $r_{0}$ is solved, the locations of $x, y$, and $z$ can be solved. The solution of $r_{0}$ is given below.
Firstly substitute expression (4.35) into the third formula in expression (4.32), which yields

$$
\begin{gather*}
\left(a_{1}^{2}+a_{2}^{2}\right) r_{0}^{2}+\left(b_{1}^{2}+b_{2}^{2}+\frac{1}{\left(1-e^{2}\right)^{2}}\right) z^{2}+2\left(a_{1} b_{1}+a_{2} b_{2}\right) r_{0} z+2\left(a_{1} c_{1}+a_{2} c_{2}\right) r_{0} \\
+2\left(b_{1} c_{1}+b_{2} c_{2}\right) z+c_{1}^{2}+c_{2}^{2}-a^{2}=0 \tag{4.43}
\end{gather*}
$$

Replacing the variable in the expression above yields

$$
\begin{equation*}
n_{1} r_{0}^{2}+n_{2} z^{2}+n_{3} r_{0} z+n_{4} r_{0}+n_{5} z+n_{6}=0 \tag{4.44}
\end{equation*}
$$

where

$$
\left.\begin{array}{l}
n_{1}=a_{1}^{2}+a_{2}^{2} \\
n_{2}=b_{1}^{2}+b_{2}^{2}+\frac{1}{\left(1-e^{2}\right)} \\
n_{3}=2\left(a_{1} b_{1}+a_{2} b_{2}\right) \\
n_{4}=2\left(a_{1} c_{1}+a_{2} c_{2}\right) \\
n_{5}=2\left(b_{1} c_{1}+b_{2} c_{2}\right) \\
n_{6}=c_{1}^{2}+c_{2}^{2}-a^{2}
\end{array}\right\} .
$$

Substituting expression (4.41) into expression (4.44) yields

$$
\begin{align*}
& \left(n_{1}+n_{2} D\right) r_{0}^{2}+\left(n_{4}+n_{2} C+n_{3} A\right) r_{0} \pm n_{3} r_{0} \sqrt{B+C r_{0}+D r_{0}^{2}} \\
& \quad \pm\left(2 n_{2} A+n_{5}\right) \sqrt{B+C r_{0}+D r_{0}^{2}}+n_{6}+n_{2}\left(A^{2}+B\right)+n_{5} A=0 \tag{4.45}
\end{align*}
$$

Replacing the parameter yields

$$
\begin{equation*}
r_{0}^{2}+m_{1} r_{0}+m_{2} r_{0} \sqrt{B+C r_{0}+D r_{0}^{2}}+m_{3} \sqrt{B+C r_{0}+D r_{0}^{2}}+m_{4}=0 \tag{4.46}
\end{equation*}
$$

where

$$
\left.\begin{array}{l}
m_{1}=\frac{n_{4}+n_{2} C+n_{3} A}{n_{1}+n_{2} D} \\
m_{2}= \pm \frac{n_{3}}{n_{1}+n_{2} D} \\
m_{3}= \pm \frac{2 n_{2} A+n_{5}}{n_{1}+n_{2} D} \\
m_{4}=\frac{n_{6}+n_{2}\left(A^{2}+B\right)+n_{5} A}{n_{1}+n_{2} D}
\end{array}\right\}
$$

Translate expression (4.46):

$$
r_{0}^{2}+m_{1} r_{0}+m_{4}=\left(-m_{2} r_{0}-m_{3}\right) \sqrt{B+C r_{0}+D r_{0}^{2}}
$$

Square both sides of the expression and simplify it to get a quartic equation:

$$
\begin{equation*}
r_{0}^{4}+s_{1} r_{0}^{3}+s_{2} r_{0}^{2}+s_{3} r_{0}+s_{4}=0 \tag{4.47}
\end{equation*}
$$

The parameters are

$$
\left.\begin{array}{l}
s_{1}=\frac{2 m_{1}-\left(m_{2}^{2} C+2 m_{2} m_{3} D\right)}{1-m_{2}^{2} D} \\
s_{2}=\frac{m_{1}^{2}+2 m_{4}-\left(m_{2}^{2} B+2 m_{2} m_{3} C+m_{3}^{2} D\right)}{1-m_{2}{ }^{2} D} \\
s_{3}=\frac{2 m_{1} m_{4}-\left(m_{3}^{2} C+2 m_{2} m_{3} B\right)}{1-m_{2}^{2} D} \\
s_{4}=\frac{m_{4}^{2}-m_{3}^{2} B}{1-m_{2}^{2} D}
\end{array}\right\} .
$$

Solve the quartic equation (Equation (4.47)) to obtain $r_{0}$, and then substitute it into expression (4.42) to solve ( $x, y, z$ ). In fact, eight groups of solutions can be obtained by using the above method to solve the expression. Only four groups conform to the system of geolocation expression (4.32), while the other four groups are the extraneous roots generated in the expression solving process. In the four groups of geolocation solutions, only one group is the location of the emitter and the others are the imaginary roots. Therefore, operations of root verification and ambiguity solving are required.

### 4.2.2 Spherical Iteration Method

Under the spherical model, the algebraic method is discussed in Section 4.2.1. Therefore, the earth ellipsoid expression (4.31) in geolocation expressions may be modified into a spherical iteration scheme, in order to implement an iterative calculation under the WGS-84 ellipse earth model (see Figure 4.5 for this concept). Firstly, use a sphere with a radius as the geocentric radius vector of a known point $\left(\mathrm{A}_{0}\right)$ to approximate the earth ellipsoid and obtain a locating point $\left(B_{1}\right)$, use the spherical algebraic algorithm and project coordinates of point $\left(B_{1}\right)$ on the earth ellipsoid surface to get $\left(\mathrm{A}_{1}\right)$, and then use the radius vector of such a point to approximate the earth's surface as the radius to obtain locating points $\left(\mathrm{B}_{2}\right)$ and $\left(\mathrm{A}_{2}\right)$. Make successive calculations to obtain project points $\left(\mathrm{A}_{3}\right),\left(\mathrm{A}_{4}\right), \ldots$ to gradually approach the intersection point $(\mathrm{J})$ of the ellipsoid and TDOA surface. Convergence of this iteration method is analyzed as follows.
For the final convergence of iteration, $\left\|\mathrm{JA}_{k}\right\|>\left\|\mathrm{JA}_{k+1}\right\|$ will be satisfied according to the convergence requirements. Since the curvature radius of the earth ellipsoid and geocentric vector radius sphere adjacent to point ( J ) is quite large and the location bias is very small compared with the satellite height and earth radius, the straight line and plane can be approximately used to replace the intersection line of the TDOA surface and the earth's surface. In triangle $\Delta \mathrm{JA}_{\mathrm{k}+1} \mathrm{~B}_{\mathrm{k}+1},\left\|\mathrm{JA}_{k+1}\right\| \leq\left\|\mathrm{JB}_{k+1}\right\|$, so only if $\left\|\mathrm{JA}_{k}\right\|>\left\|\mathrm{JB}_{k+1}\right\|$ are


Figure 4.5 Iteration principle of the spherical iteration method
the convergence conditions wholly satisfied. In triangle $\Delta \mathrm{JA}_{k} \mathrm{~B}_{k+1}$, if $\left\|J A_{k}\right\|>\left\|J B_{k+1}\right\|$, $\angle \mathrm{JA}_{k} \mathrm{~B}_{k+1}<\angle \mathrm{JB}_{k+1} \mathrm{~A}_{k}$, so inclination between the intersection line of the TDOA hyperboloid and the earth ellipsoid is $\angle \mathrm{B}_{k} \mathrm{JA}_{k}>2 \angle \mathrm{JA}_{k} \mathrm{~B}_{k+1}$, while the projected angle $\angle \mathrm{JA}_{k} \mathrm{~B}_{k+1}$ is smaller than angle $\delta(<4 \mathrm{mrad})$ between the regular sphere and the earth ellipsoid. Therefore, only if the inclination $\angle \mathrm{B}_{k} \mathrm{JA}_{k}>2 \delta$ can iterative convergence be ensured; obviously, it is possible to meet this condition. However, it cannot be achieved when the intersection line of the TDOA hyperboloid is tangent to the earth ellipsoid. As the distance between satellites is far smaller than the height of the satellite, the TDOA surface where the emitter is under subsatellite coverage will the earth's surface inevitably intersect at a large inclination. Therefore, only if the expressions can be solved can the condition $\angle \mathrm{B}_{k} \mathrm{JA}_{k}>2 \delta$ be satisfied. According to the forgoing, it is concluded that the iteration method is effective in the geolocation area.
Suppose location $\mathrm{B}_{k}\left(x^{(k)}, y^{(k)}, z^{(k)}\right)$ of the emitter is obtained via $k$ iterations. Suppose location $\mathrm{B}_{k+1}\left(x^{(k+1)}, y^{(k+1)}, z^{(k+1)}\right)$ of the emitter can be obtained via $k+1$ iterative calculations:

Step 1. Use expressions (2.32) to (2.37). Conduct coordinate transformation to the location of the emitter $\mathrm{B}_{k}\left(x^{(k)}, y^{(k)}, z^{(k)}\right)$ and work out the previous time to obtain longitude, latitude, and altitude coordinates of $\mathrm{B}_{k}\left(L^{(k)}, B^{(k)}, H^{(k)}\right)$. Then use the projecting location coordinate of such a point on the earth ellipsoid surface to find $\mathrm{A}_{k}\left(L^{(k)}, B^{(k)}, 0\right)$. Use expression (2.30) to transform the longitude, latitude, and altitude coordinates of $\mathrm{A}_{k}$ to geocentric Cartesian coordinates $\mathrm{A}_{k}\left(x_{p}{ }^{(k)}, y_{p}{ }^{(k)}, z_{p}{ }^{(k)}\right)$.
Step 2. Approximate the sphere with a location ellipsoid surface taking the geocentric radius vector of $\mathrm{A}_{k}\left(x_{p}{ }^{(k)}, y_{p}{ }^{(k)}, z_{p}{ }^{(k)}\right)$ as the radius:

$$
\begin{equation*}
x^{(k+1) 2}+y^{(k+1) 2}+z^{(k+1) 2}=x_{p}^{(k) 2}+y_{p}^{(k) 2}+z_{p}^{(k) 2} \tag{4.48}
\end{equation*}
$$

The point $\mathrm{A}_{k}\left(x_{p}{ }^{(k)}, y_{p}{ }^{(k)}, z_{p}{ }^{(k)}\right)$ on the earth ellipsoid satisfies

$$
\begin{equation*}
x_{p}^{(k) 2}+y_{p}^{(k) 2}+\frac{1}{1-e^{2}} z_{p}^{(k) 2}=a^{2} \tag{4.49}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
x^{(k+1) 2}+y^{(k+1) 2}+z^{(k+1) 2}=a^{2}+\left(1-\frac{1}{1-e^{2}}\right) z_{p}^{(k) 2} \triangleq R^{(k) 2} . \tag{4.50}
\end{equation*}
$$

Taking the first expression of Equation (4.2) from expression (4.50) yields

$$
\begin{equation*}
x_{0} x^{(k+1)}+y_{0} y^{(k+1)}+z_{0} z^{(k+1)}=k_{3}^{\prime}-\frac{1}{2} r_{0}{ }^{(k+1)^{2}} \tag{4.51}
\end{equation*}
$$

where

$$
\begin{aligned}
k_{3}^{\prime} & =\frac{1}{2}\left[R^{(k)^{2}}+\left(x_{0}^{2}+y_{0}^{2}+z_{0}^{2}\right)\right], \\
r_{0}{ }^{(k+1)^{2}} & =\left(x^{(k+1)}-x_{0}\right)^{2}+\left(y^{(k+1)}-y_{0}\right)^{2}+\left(z^{(k+1)}-z_{0}\right)^{2},
\end{aligned}
$$

and $r_{0}{ }^{(k+1)}$ is the distance between the $(k+1)$ th iterative location point $\mathrm{B}_{k+1}\left(x^{(k+1)}, y^{(k+1)}\right.$, $z^{(k+1)}$ ) and the primary satellite, serving as an intermediate variable to solve expressions.
Step 3. Combine the first two expressions in Equations (4.32) and (4.51) and use the spherical mode expression solution method indicated in expressions (4.7) to (4.11) (replace $k_{3}$ with $k_{3}{ }^{\prime}$ ) to solve the location of the emitter $\mathrm{B}_{k+1}\left(x^{(k+1)}, y^{(k+1)}, z^{(k+1)}\right)$. Define $d^{(k)}=\sqrt{\left(x_{p}^{(k)}-x_{p}^{(k-1)}\right)^{2}+\left(y_{p}^{(k)}-y_{p}^{(k-1)}\right)^{2}+\left(z_{p}^{(k)}-z_{p}^{(k-1)}\right)^{2}}$ and stop the iteration in the case of $d^{(k)}<\varepsilon$, thereby obtaining the accurate location of the emitter.

The initial location result of the regular spherical model set out in Section 4.1.1 is used. To reduce excessive iteration caused by a large initial error, $R$ will be approximated to the geoidal altitude of the emitter. Assuming that the location of the satellite is available, the geocentric radius vector range of the subsatellite point of the primary satellite (or auxiliary satellite) can be deemed to be the initial radius $R$ of the location sphere.

For the projection process in Step 1, as an iterative calculation is required when geocentric Cartesian coordinates are transformed to longitude, latitude, and altitude coordinates, the computational load is relatively heavy. To reduce the operation workload, considering that the altitude $H$ is subject to location is very small compared with the radius of the earth, zero-altitude deformation can be used to implement an approximate projection to expressions (4.7) to (4.11). This is known as the simplified spherical iteration algorithm, with expressions as follows:

$$
\begin{align*}
& B^{(k)}=\tan ^{-1}\left(\frac{z^{(k)}}{\left.\left(1-e^{2}\right) \sqrt{x^{(k)^{2}+y^{(k)^{2}}}}\right)},\right.  \tag{4.52}\\
& L^{(k)}=\tan ^{-1}\left(y^{(k)} / x^{(k)}\right) . \tag{4.53}
\end{align*}
$$

### 4.2.3 Newton Iteration Method

The Newton iteration method of nonlinear expressions is a typical iteration method, which is proposed based upon the concept of successive linearization to nonlinear expressions. Generally speaking, Newton iteration features fast convergence speed, but it is required to change differential quotients to nonlinear functions.

Consider the nonlinear expressions

$$
\begin{equation*}
\mathbf{F}(\mathbf{x})=\mathbf{0} . \tag{4.54}
\end{equation*}
$$

Let $x^{*}$ be the solution of this expression; then $\mathbf{x}^{(k)}$ is the approximate solution. If $\mathbf{F}(\mathbf{x})$ is differentiable near $\mathbf{x}^{(k)}$, the $\mathbf{F}(\mathbf{x})$ can be linearized near $\mathbf{x}^{(k)}$ to obtain the approximate linear expression:

$$
\begin{equation*}
\mathbf{F}(\mathbf{x})=\mathbf{F}\left(\mathbf{x}^{(k)}\right)+\mathbf{F}^{\prime}\left(\mathbf{x}^{(k)}\right)\left(\mathbf{x}-\mathbf{x}^{(k)}\right)=0 \tag{4.55}
\end{equation*}
$$

When $\mathbf{F}^{\prime}\left(\mathbf{x}^{(k)}\right)$ is nonsingular, the expression (4.54) has only one solution, that is, $\mathbf{x}^{(k+1)}$. In this way, the Newton iteration scheme [11] is obtained:

$$
\begin{equation*}
\mathbf{x}^{(k+1)}=\mathbf{x}^{(k)}-\left[\mathbf{F}^{\prime}\left(\mathbf{x}^{(k)}\right)\right]^{-1} \mathbf{F}\left(\mathbf{x}^{(k)}\right) . \tag{4.56}
\end{equation*}
$$

Rewrite the location expression (4.40) as follows:

$$
\left.\begin{array}{l}
f_{1}=\sqrt{\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}+\left(z-z_{1}\right)^{2}}-\sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2}}-c \Delta t_{1} \\
f_{2}=\sqrt{\left(x-x_{2}\right)^{2}+\left(y-y_{2}\right)^{2}+\left(z-z_{2}\right)^{2}}-\sqrt{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2}}-c \Delta t_{2} \\
f_{3}=\frac{x^{2}}{a}+\frac{y^{2}}{a}+\frac{z^{2}}{\left(1-e^{2}\right) a}-a \tag{4.57}
\end{array}\right\} .
$$

Let $\mathbf{x}=\left[\begin{array}{ll}x & y \\ z\end{array}\right]^{\mathrm{T}}$ and $\mathbf{F}(\mathbf{x})=\left[\begin{array}{ll}f_{1} & f_{2}\end{array} f_{3}\right]^{\mathrm{T}}$. Then

$$
\begin{align*}
\mathbf{F}^{\prime}(\mathbf{x}) & =\left[\begin{array}{lll}
\frac{\partial f_{1}}{\partial x} & \frac{\partial f_{1}}{\partial y} & \frac{\partial f_{1}}{\partial z} \\
\frac{\partial f_{2}}{\partial x} & \frac{\partial f_{2}}{\partial y} & \frac{\partial f_{2}}{\partial z} \\
\frac{\partial f_{3}}{\partial x} & \frac{\partial f_{3}}{\partial y} & \frac{\partial f_{3}}{\partial z}
\end{array}\right] \\
& =\left[\begin{array}{ccc}
\frac{x-x_{1}}{r_{1}}-\frac{x-x_{0}}{r_{0}} & \frac{y-y_{1}}{r_{1}}-\frac{y-y_{0}}{r_{2}}-\frac{z-z_{1}}{r_{0}}-\frac{z-z_{0}}{r_{1}} & \frac{y-y_{2}}{r_{2}}-\frac{y-y_{0}}{r_{0}} \\
\frac{z-z_{2}}{r_{2}}-\frac{z-z_{0}}{r_{0}} \\
2 x / a & 2 y / a & \frac{2 z}{\left(1-e^{2}\right) a}
\end{array}\right] . \tag{4.58}
\end{align*}
$$

Substitute Equation (4.58) into the iteration scheme (Equation (4.56)) to implement successive iterations and stop the iterations when $\left\|x^{(k)}-x^{(k-1)}\right\|<\varepsilon$ is achieved.
With respect to the Newton iteration method, matrix inversion $\left[\mathbf{F}^{\prime}\left(\mathbf{x}^{(k)}\right)\right]^{-1}$ is required to be calculated for each iteration, which may cause a heavy computational load. To reduce the computational load, a simplified Newton iteration method may be employed, that is, always take $\mathbf{F}^{\prime}\left(\mathbf{x}^{(k)}\right)$ to be $\mathbf{F}^{\prime}\left(\mathbf{x}^{(0)}\right)$ during the iteration process:

$$
\begin{equation*}
\mathbf{x}^{(k+1)}=\mathbf{x}^{(k)}-\left[\mathbf{F}^{\prime}\left(\mathbf{x}^{(0)}\right)\right]^{-1} \mathbf{F}\left(\mathbf{x}^{(k)}\right) \tag{4.59}
\end{equation*}
$$

After this, the computational load will decrease while the convergence speed is decreasing and only the linear convergence speed is implemented.
Similarly, the initial location result of the spherical model set out in Section 4.1.1 is used to estimate the value of the initial iteration point.

Table 4.1 Operation time required by the three expression solution methods

|  |  | Initial operation time (s) | Ambiguity resolution time (s) | Average iteration time (s) | Iteration time (s) | Operation time (s) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Analytical algorithm |  | 46.5640 | 22.0650 | 0 | 0 | 68.6290 |
| Spherical iteration method | $\varepsilon=1 \mathrm{~m}$ | 41.6380 | 15.6060 | 2.2269 | 204.3630 | 261.6070 |
|  | $\varepsilon=0.001 \mathrm{~m}$ | 41.5130 | 15.6440 | 3.1423 | 272.0520 | 329.2090 |
| Simplified spherical iteration method | $\varepsilon=1 \mathrm{~m}$ | 41.1260 | 15.5560 | 2.2000 | 99.6840 | 156.3660 |
|  | $\varepsilon=0.001 \mathrm{~m}$ | 41.1940 | 15.4120 | 3.1423 | 144.6350 | 195.8910 |
| Newton iteration method | $\varepsilon=1 \mathrm{~m}$ | 40.9080 | 15.3410 | 2.0654 | 45.7690 | 102.0180 |
|  | $\varepsilon=0.001 \mathrm{~m}$ | 40.9470 | 15.4210 | 2.8615 | 62.6220 | 118.9900 |
| Simplified Newton iteration method | $\varepsilon=1 \mathrm{~m}$ | 41.0640 | 15.3770 | 2.0654 | 37.9660 | 94.4070 |
|  | $\varepsilon=0.001 \mathrm{~m}$ | 41.0300 | 15.3630 | 2.9846 | 50.7620 | 107.1550 |

### 4.2.4 Performance Comparison among the Three Solution Methods

We introduced three location methods based on the WGS-84 earth ellipsoid model. A comparison among the three methods will be given below, focusing on computing time and numerical error implemented by computer and algorithm implementation conditions.

### 4.2.4.1 Computing of Three Solution Methods

The given locations of the satellites are $\mathrm{O}_{0}\left(125.765^{\circ} \mathrm{E}, 9.87^{\circ} \mathrm{N}\right), \mathrm{O}_{1}\left(126.166^{\circ} \mathrm{E}, 10.54^{\circ} \mathrm{N}\right)$, and $\mathrm{O}_{2}\left(126.54^{\circ} \mathrm{E}, 9.87^{\circ} \mathrm{N}\right)$, the distance between the satellites is about 100 km , the height of the satellites is 1000 km , and the minimum observation angle $\sigma=25^{\circ}$. The aim is to carry out geolocation calculations to the same quantity of points on the earth's surface from nearby subsatellites without noise or supposed altitude error; the consumed time is as shown in Table 4.1. Supposing that geolocation ambiguity can be solved correctly, and each iteration method starts with a spherical model, the iterative calculation is implemented after the ambiguity resolution is implemented, where $\varepsilon$ is the termination condition of iterative convergence.
From Table 4.1, some conclusions can be made:

1. Comparatively speaking, the solution process of the analytical method based on the WGS-84 earth ellipsoid model required a heavier computational load and longer ambiguity resolution time than that required by the initial location solution via a regular spherical model. However, the former method only requires a small computational load because the iterative calculation is not required. Therefore it features the smallest computational load and fastest speed.
2. Among all iteration processes, the Newton iteration method produces the fastest convergence speed and the least iterative times, but its computing time is longer than the analytical method although far superior than the spherical iteration method. The simplified Newton iteration method has increased iterations but because only one time of the partial differential matrix is calculated, its computational load is less than the Newton iteration method.
3. Iterative convergence is time consuming when the spherical iteration method is used. Its operational consumption includes: firstly, a regular spherical geolocation calculation is required for each iteration, during which a quartic equation is required to be solved, so the computational load is heavy; secondly, geocentric Cartesian coordinates need to be transformed into longitude, latitude, and altitude coordinates during each iterative projection process, which requires implementation of iteration. According to the foregoing, the spherical iteration method has the slowest computation speed. A simplified spherical iteration method uses zero-altitude to approximately simplify the projection process and causes projection error, which will not create obvious influence under small altitudes, so the computational load is reduced and the projection time shortened.

In a word, the analytical method enjoys the fastest computation speed, followed by the Newton iteration method, with the spherical iteration algorithm having the slowest speed.

### 4.2.4.2 Numerical Error Simulation

When a computer or hardware equipment is used to implement the foresaid algorithms, numerical error will inevitably be caused by a limited accuracy finite word length, which is one of the important constraints of implemention by a control algorithm. Obviously, numerical accuracy of an analytical calculation is only limited by finite word length, but numerical accuracy of an iteration method is correlated to iterative convergence conditions.
Suppose the constellation parameters of the satellite are as follows: the orbit height of the primary satellite is 1000 km , the eccentricity is 0 (elliptical orbit), the orbit inclination is $60^{\circ}$, the right ascension of the ascending node (RAAN) is $120^{\circ}$, the argument of perigee is $0^{\circ}$, the time of perigee is 0 second, the same orbit auxiliary satellite is 100 km ahead of the primary satellite, and the ascending node of the satellite not on the same track moves 100 km to the east. Suppose there are no bias and errors in the TDOA measurement, the numerical calculation error investigated is from $t=300$ to 700 seconds for the three algorithms, among which the data are expressed with the double data format, and $\varepsilon=10^{-6}$ is taken as the termination condition of the iteration method. The maximum error value curve of the subsatellite point location of the primary satellite is shown in Figure 4.6.
It is shown from Figure 4.6 that the analytical method is influenced greatly by the numerical quantization error. There is a centimeter-level numerical error even in absence of the TDOA measurement error. Although such an error is relatively small compared with the geolocation error caused by bias and the random measurement error, it indicates that numerical stability of the analytical method is not as good as the iteration method, which will arouse attention during practical application.


Figure 4.6 Numerical error quantity of the three methods. Numerical error of (a) the analytical method, (b) the spherical iteration method, and (c) the Newton iteration algorithm

(c)

Figure 4.6 (continued)

### 4.2.4.3 Algorithm Implementation Conditions

Each of the geolocation solution methods above has its own application condition, which determines its application scope and serves as an important index of algorithm efficiency. In the analytical method, to get the parameter substitution expression (4.36), we propose the factor

$$
J=\left(x_{0}-x_{1}\right)\left(y_{0}-y_{2}\right)-\left(x_{0}-x_{2}\right)\left(y_{0}-y_{1}\right) \neq 0
$$

for the denominator. This is an application condition of the analytical algorithm. The square root of the expression (4.41) is used to solve $z$ and an imaginary number will theoretically be created; in fact, however, the location surface can intersect at least one point if there is no observation noise. Therefore, $z$ must have a real number solution. If the location surface cannot intersect due to noise influence, this would indicate that accuracy of the system at this point is poor. No restriction is demanded by this geolocation solution method.
The spherical iteration method is mainly composed of regular special model algorithms. Therefore, its application conditions are the same as the spherical model algorithm, that is, matrix $\mathbf{A}$ in expression (4.7) should be invertible. When $\mathbf{A}$ is not invertible, such an iteration method cannot be implemented. For the Newton iteration method, the Jacobi matrix is generally invertible (the condition number of the Jacobi matrix is large when a three-satellite projection approaches collineation and the emitter is located on such a line). If the optimum initial value can be obtained, the application conditions are the same as those of the spherical iteration method.
According to the above, the factor $J$ and $\operatorname{cond}(\mathbf{A})$ can be used to express application conditions of the analytical method and the iteration method, respectively.

Figure 4.7 shows the fluctuation curve of $J$ and $\operatorname{cond}(\mathbf{A})$ under the circumstance where the constellation orbits the earth for a complete revolution; in this case, orbiting the earth for a complete revolution, the satellite will spend

$$
T=2 \pi \sqrt{\frac{(a+H)^{3}}{\mu}} \approx 6307 \text { seconds }
$$

where $\mu$ is the gravitational constant. For the three expression solution methods, commencing from $t=-100$ seconds and ending on $t=6300$ seconds, take the three-satellite location at an interval of 1 second to calculate $J$ and $\operatorname{cond}(\mathbf{A})$ and one can obtain Figure 4.7.
According to Figure 4.7, during one cycle of the constellation orbits, $J$ crosses the zero point four times but $\log _{10} \operatorname{cond}(\mathbf{A})$ tends to be infinity for two times and corresponds to two zero-point locations of $J$. As the location of the two infinity points of $\log _{10} \operatorname{cond}(\mathbf{A})$ corresponds to the system's weak observation condition, the other two zero-points of $J$ are introduced during the expression solution. Therefore, application conditions of the analytical method are more rigorous than those of the iteration method, which damages the system's geolocating capacity to some extent.
According to the above, it can be concluded that the analytical method is fast in computing speed but may cause blind areas of geolocation where there should be optimum geolocation areas according to further research. Therefore the analytical method is not ideal. The iteration method maintains the system's locating capacity but requires heavier computational load and slow computational speed. Therefore, the foregoing two methods can be combined, that is, the analytical method is used in general circumstances and the simplified Newton iteration method can be used when the constellation enters the blind area of the analytical method, in order to get fast computational speed and high accuracy.

### 4.2.5 Altitude Input Location Algorithm

When the location algorithm based on the WGS-84 earth ellipsoid surface was discussed in the section above, a zero-altitude assumption is used to calculate data for the emitter and is effective for sea emitters and low-altitude emitters, but large bias may be introduced for high-altitude emitters. To minimize bias, several methods like the geographical information system (GIS) can be used to provide approximate altitude data of the emitter, in order to locate the emitter accurately under altitude input.
For application of the altitude input location algorithm, the following aspects may be considered: firstly, use the zero-altitude assumption method to roughly locate the emitter and analyze the location of the locating point. If the altitude on which the emitter point is located cannot approximate to zero, use the GIS to find the altitude data. If the altitude fluctuates mildly, such a method is effective in making an altitude estimation. Lastly, utilize the estimated altitude to implement the altitude input location to the emitter. The altitude input algorithm is discussed below.
When the emitter has altitude $\mathrm{d} H$, rewrite the earth surface equation on which the emitter is located:

$$
\begin{equation*}
\frac{x^{2}}{(N+\mathrm{d} H)^{2}}+\frac{y^{2}}{(N+\mathrm{d} H)^{2}}+\frac{z^{2}}{\left[N\left(1-e^{2}\right)+\mathrm{d} H\right]^{2}}=1 \tag{4.60}
\end{equation*}
$$



Figure 4.7 Fluctuation of $J$ and $\operatorname{cond}(\mathbf{A})$ with time: (a) curve of $J$ and (b) curve of $\log _{10} \operatorname{cond}(\mathbf{A})$
where $N$ is the curvature radius of the local prime vertical circle and a function of the emitter location coordinate. Combine the expression above and expression (4.3), which is the three-satellite geolocation expression taking the input altitude ellipsoid surface as the geolocation plane.
Obviously, Equation (4.60) is more completed than the earth ellipsoid surface equation (Equation (4.31)). It is not a regular ellipsoid surface, so it is difficult to establish analytical methods and the Newton iteration method. Therefore, the spherical iteration method is considered, with calculation procedures as follows:

Step 1. Use the location algorithm under the zero-altitude assumption to get an initial geolocation $\mathrm{B}_{0}\left(x^{(0)}, y^{(0)}, z^{(0)}\right)$ and transform it into longitude, latitude, and altitude coordinates $\mathrm{B}_{0}\left(L^{(0)}, B^{(0)}, 0\right)$.
Step 2. Suppose the altitude of the emitter is $\mathrm{d} H$. Project $\mathrm{B}_{k}\left(L^{(k)}, B^{(k)}, 0\right)$ to the location on the earth's surface with $\mathrm{d} H$ of altitude, that is, $\mathrm{A}_{k}\left(L^{(k)}, B^{(k)}, d H\right)$, and transform it into Cartesian coordinates $\mathrm{A}_{k}\left(x_{p}^{(k)}, y_{p}^{(k)}, z_{p}^{(k)}\right)$. Then calculate the curvature radius of the prime vertical circle of a point such as $N^{(k)}$.
Step 3. Take the approximated sphere of the earth's surface, taking the geocentric radius vector of location $\mathrm{A}_{k}\left(x_{p}^{(k)}, y_{p}^{(k)}, z_{p}^{(k)}\right)$ as the radius:

$$
\begin{equation*}
x^{(k+1) 2}+y^{(k+1) 2}+z^{(k+1) 2}=x_{p}^{(k) 2}+y_{p}^{(k) 2}+z_{p}^{(k) 2} \tag{4.61}
\end{equation*}
$$

Substitute surface expression (4.60), which met at point $\mathrm{A}_{k}\left(x_{p}^{(k)}, y_{p}^{(k)}, z_{p}^{(k)}\right)$, into expression (4.61). Then

$$
\begin{equation*}
x^{(k+1) 2}+y^{(k+1) 2}+z^{(k+1) 2}=\left(N^{(k)}-\mathrm{d} H\right)^{2}+\left[1-\frac{\left(N^{(k)}+\mathrm{d} H\right)^{2}}{\left[N^{(k)}\left(1-e^{2}\right)+\mathrm{d} H\right]^{2}}\right] z^{(k)^{2}} \triangleq R^{(k) 2} \tag{4.62}
\end{equation*}
$$

Use expression (4.31) and expressions (4.33) to (4.47) to find the locating point of emitter $\mathrm{B}_{k+1}\left(x^{(k+1)}, y^{(k+1)}, z^{(k+1)}\right)$.
Step 4. Repeat Steps 2 and 3, until

$$
d^{(k)}=\sqrt{\left(x_{p}^{(k)}-x_{p}^{(k-1)}\right)^{2}+\left(y_{p}^{(k)}-y_{p}^{(k-1)}\right)^{2}+\left(z_{p}^{(k)}-z_{p}^{(k-1)}\right)^{2}}<\varepsilon .
$$

Then the output result is the location of the emitter under the altitude assumption.

### 4.3 Ambiguity and No-Solution Problems of Geolocation

### 4.3.1 Ambiguity Problem of Geolocation

In the three-satellite geolocation system, we employ two TDOA geolocation surfaces and the earth's surface to solve the location of the emitter. In terms of a physical concept, TDOA geolocation surfaces determined by two independent TDOA intersect on a quadric curve, the intersection point of the curve and the earth surface being the location of the emitter. Since the earth's surface is a closed surface, the TDOA quadric curve must intersect the earth's surface at two points. Therefore, if a solution exists, the three-satellite geolocation system is always challenged by the geolocation ambiguity problem. This is shown in Figure 4.8.
According to Figure 4.8, the intersection line of general TDOAs always intersects the earth's surface at two points ( $\mathrm{J}_{0}$ and $\mathrm{J}_{1}$ ). In most circumstances, the two intersection points are not all located in the threesatellite coverage area at the same time. Therefore, the ambiguity problem can be solved by determining whether the locating point is located within the satellite coverage area. The process falls into two parts:

1. Delete the intersection point of the TDOA curve at the other hemisphere, that is, calculate the distance between the locating point and the satellite. If the distance is larger than the sum of the satellite height and the earth radius, the locating point is located on the other hemisphere of the earth and it should be deleted.
2. Calculate the intersection angle between the location coordinate point and each satellite subsatellite point, and observe whether such an angle is smaller than the coverage angle ( $\theta$ ) of the satellite against the earth; if not, the locating point is not within the coverage area of the satellite and it must be the ambiguous point.

This ambiguity solution judge process requires calculating different reference parameters. As it is complicated, simplification can be implemented:

1. The distance between satellites is much smaller compared with the satellite height, so investigation of all three satellites can be simplified to an investigation of the reference index of the primary satellite. This slightly decreases the success rate of ambiguity resolution but greatly reduces the computational load.
2. Whether the emitter is located within the satellite coverage area can be judged by the distance $r_{0}$ of the emitter from the satellite, which is subject to a limitation. Under the spherical model, draw a conical surface with coverage angle $\theta$ of the satellite; the emitter point must be within the satellite coverage area when $r_{0}$ is smaller than the length of the generatrix of the conical surface. Under the ellipse model, such an approximation will cause a slight change of the ambiguity resolution performance. However, considering that the system does not be precise to the point at the coverage area edge, the influence caused by such a change is not serious and can be mitigated by overlapping the design to the coverage area.

By ambiguity resolution methods above, the ambiguous point can be effectively deleted. Figure 4.9 shows the distribution of normalized residual ambiguity under different satellite geometric configuration conditions after the foregoing ambiguity resolution processing is implemented. Suppose the geometric configuration of the satellite constellation is as shown in Figure 4.8, with $\mathrm{O}_{1}$ and $\mathrm{O}_{2}$ fixed. Take the perpendicular bisector of the line between $\mathrm{O}_{1}$ and $\mathrm{O}_{2}$ as the xaxis and the connecting line as the yaxis. The location of $\mathrm{O}_{3}$ changes within the rectangular area, as indicated in the Figure 4.9.


Figure 4.8 Ambiguity problem of the three-satellite geolocation


Figure 4.9 Schematic diagram of the constellation geometric configuration


Figure 4.10 Residual ambiguity area distribution under different constellation geometric configuration conditions

Suppose the location of $\mathrm{O}_{0}, \mathrm{O}_{1}$ is fixed and $\mathrm{O}_{2}$ changes in the rectangular area between $\mathrm{O}_{0}$ and $\mathrm{O}_{1}$, and the satellite height is 1000 km . The distribution of residual ambiguity under different satellite constellation geometric configuration conditions is as shown in Figure 4.10, according to which, most of the three-satellite constellation structures has no residual ambiguity after ambiguity resolution of the coverage area. The residual ambiguity area will increase rapidly only when the projection of all satellites is in approximately a straight line.


Figure 4.11 Fluctuation of the residual ambiguity area and $\log _{10}(\operatorname{cond}(\mathbf{A}))$ when the satellite orbits the earth for a complete revolution

The constellation structure is always changing when satellites are orbiting the earth. It is therefore necessary to research the fluctuation rule of the residual ambiguity area when a satellite orbits the earth for a complete revolution. Figure 4.11 provides the fluctuation rule of the normalized residual ambiguity area and $\log _{10}(\operatorname{cond}(\mathbf{A}))$ when the satellite orbits the earth for a complete revolution after the foregoing ambiguity resolution method is used. The constellation parameters of the satellite are the same as given above, and the minimum observation angle of the satellite is $\varepsilon=25^{\circ}$.
It is revealed from Figure 4.11 that the satellite's coverage area has no geolocation ambiguity area against the emitter location under the subsatellite coverage area during most of the time when the satellite orbits the earth for a complete revolution after ambiguity resolution processing. There is residual ambiguity of geolocation only within a small area (simulation indicates that the period during which the ambiguity exists is less than $4 \%$ o). According to this figure, the constellation is within a weak observation period while the residual ambiguity exists. In fact, the constellation is incapable of geolocating ground emitters within this period, so it is meaningless to discuss the geolocation ambiguity problem in this situation.
The foregoing research reveals that the ambiguity problem of the three-satellite geolocation system can be effectively resolved in most cases.


Figure 4.12 Schematic diagram of the no-solution problem of a three-satellite geolocation system

### 4.3.2 No-Solution Problem of Geolocation

There are always noise interference and measurement error in the TDOA geolocation system, which may cause deviation of the TDOA hyperboloid. In terms of algebra, such a deviation will cause the three location surfaces to intersect not at a common point but in a small area. As shown in Figure 4.12, the intersection lines of two TDOA hyperboloids located above the earth's surface do not intersect. In this case, location expressions have no real solution.
The no-solution problem is caused by observation error but is restricted by the satellite constellation geometric configuration. Figure 4.13a shows the normalized no-solution area distribution when there is no solution in the subsatellite area under different satellite geometric configuration conditions (the same as Figure 4.9), where the TDOA measurement RMS $\sigma_{\Delta t}=100 \mathrm{~ns}$, and the measurement error of the satellite position $\sigma_{s}$ is 30 m in all directions and does not correlate to each other. If the error quantity is increased to make the TDOA measurement RMS $\sigma_{\Delta t}=300 \mathrm{~ns}$ and the measurement error of satellite position $\sigma_{s}=100 \mathrm{~m}$ in all directions, the normalized no-solution area distribution is as shown in Figure 4.13b.
According to Figure 4.13, it can be concluded that:

1. The no-solution problem will arise when three satellites are approximately in the same line. At this time, the observability of the system is quite poor. There will not be a no-solution problem when observability of the three-satellite geometric structure is good. Therefore, the no-solution problem is a representation where the system is in weak observability.
2. The no-solution area may be reduced under the same conditions by improving measurement accuracy. However, such an improvement will impose little influence upon the no-solution area distribution with a geometric configuration change. Therefore, if the constellation is designed carefully, there will not be a no-solution problem. Nevertheless, the movement of the satellites will change the constellation geometric structure, so both the no-solution area and the no-solution duration should be optimized in satellite constellation design.


Figure 4.13 Influence of the constellation on the no-solution area distribution under different noise intensities: (a) $\sigma_{\Delta t}=100 \mathrm{~ns}, \sigma_{s}=30 \mathrm{~m}$ and (b) $\sigma_{\Delta t}=300 \mathrm{~ns}, \sigma_{s}=100 \mathrm{~m}$

Similarly, Figure 4.14 shows the change in the rules of statistics value of the no-solution spatial coverage of a satellite with time when the constellation orbits the earth for a complete revolution. Here, the constellation parameters are the same as in Section 4.3.1. Carry out Monte Carlo tests, repeating it 100 times, and such a point is deemed as a no-solution area if there has been a no-solution case more than 10 times ( $10 \%$ ). This figure clearly reveals the relationship between a no-solution area and observability, indicating that the time of a no-solution case accounts for a conditional number of matrix $\mathbf{A}$ in Equation (4.8).
To deal with no-solution problems in the three-satellite TDOA geolocation system, the point that minimizes the squared distance between the earth and two TDOA surfaces can be taken as the suboptimal location solution. Given the distance between a point $\mathrm{P}(x, y, z)$ on the


Figure 4.14 Fluctuation of the no-solution area when a constellation orbits the earth for a complete revolution
earth's surface and the two TDOA surfaces, respectively, is $d_{1}(x, y, z)$ and $d_{2}(x, y, z)$, the cost function is

$$
\begin{equation*}
F(x, y, z)=d_{1}^{2}(x, y, z)+d_{2}^{2}(x, y, z) . \tag{4.63}
\end{equation*}
$$

Find the partial derivative of $F(x, y, z)$ and let it be zero. Then

$$
\left.\begin{array}{l}
\frac{\partial(F(x, y, z))}{\partial(x)}=0 \\
\frac{\partial(F(x, y, z))}{\partial(y)}=0  \tag{4.64}\\
\frac{\partial(F(x, y, z))}{\partial(z)}=0
\end{array}\right\} .
$$

The solution can be considered as a suboptimal solution of the corresponding measurement value of location expressions. In this case, the appropriate point may be selected as the location of the emitter according to other rules.

### 4.4 Error Analysis of Three-Satellite Geolocation

Firstly, location accuracy index should be defined. In a three-satellite geolocation system, we focus on the geodetic distance between the estimated position of the emitter and its actual position. The influence of the earth curvature error can be omitted because it is small; therefore, we can make the geodetic distance between the estimated position of the emitter and its actual position equivalent to the horizontal bias of the located position in the local top-centric horizon coordinates system. Suppose the coordinates of the estimated position of the emitter in the local top-centric horizon coordinate system at the actual point is ( $x^{\prime}, y^{\prime}, z^{\prime}$ ). Define the 2D locating accuracy index as the horizontal distance $d=\sqrt{x^{\prime 2}+y^{\prime 2}}$.
For the WGS-84 ellipsoid earth surface model, the osculation error of it and the earth geoid is within 30 m [12] and can be omitted. Therefore, the major errors will be the bias caused by altitude assumption and the random geolocation error caused by random TDOA measurement noise; these two errors will arouse different influences. The fixed geolocation bias caused by bias should be measured by the locating distance, while the random geolocation error reflects the distribution of random location points and should be measured by the location error variance. Location accuracy under certain constellation geometric configuration conditions is also analyzed in this section.

### 4.4.1 Analysis of the Random Geolocation Error

### 4.4.1.1 Error Analysis Method

In three-satellite TDOA geolocation system, the random observation error basically includes the TDOA measurement errors $\mathrm{d} \Delta t_{1}, \mathrm{~d} \Delta t_{2}$ and the position error of each satellite ( $\mathrm{d} x_{i}, \mathrm{~d} y_{i}, \mathrm{~d} z_{i}$, $i=0,1,2)$.
In the ECEF coordinates, by differentiating the first two expressions of expression (4.32) at the emitter point $(x, y, z)$, one can obtain

$$
\begin{align*}
& \frac{x-x_{i}}{r_{i}}\left(\mathrm{~d} x-d x_{i}\right)+\frac{y-y_{i}}{r_{i}}\left(\mathrm{~d} y-\mathrm{d} y_{i}\right)+\frac{z-z_{i}}{r_{i}}\left(\mathrm{~d} z-\mathrm{d} z_{i}\right) \\
& \quad-\left[\frac{x-x_{0}}{r_{0}}\left(\mathrm{~d} x-\mathrm{d} x_{0}\right)+\frac{y-y_{0}}{r_{0}}\left(\mathrm{~d} y-\mathrm{d} y_{0}\right)+\frac{z-z_{0}}{r_{0}}\left(\mathrm{~d} z-\mathrm{d} z_{0}\right)\right]=\mathrm{d}\left(c \Delta t_{i}\right) \quad(i=1,2) \tag{4.65}
\end{align*}
$$

This can be transformed into

$$
\begin{equation*}
\left(C_{i x}-C_{0 x}\right) \mathrm{d} x+\left(C_{i y}-C_{0 y}\right) \mathrm{d} y+\left(C_{i z}-C_{0 z}\right) \mathrm{d} z=c \mathrm{~d}\left(\Delta t_{i}\right)+\mathbf{u}_{i} \cdot \mathbf{d}_{i}-\mathbf{u}_{0} \cdot \mathbf{d}_{0} \tag{4.66}
\end{equation*}
$$

where

$$
\begin{aligned}
C_{i x} & =\left(x-x_{i}\right) / r_{i}, C_{i y}=\left(y-y_{i}\right) / r_{i}, C_{i z}=\left(z-z_{i}\right) / r_{i}, \\
\mathbf{u}_{i} & =\left[\begin{array}{ll}
C_{i x} & C_{i y} \\
C_{i z}
\end{array}\right], \\
\mathbf{d}_{i} & =\left[\begin{array}{lll}
\mathrm{d} x_{i} & \mathrm{~d} y_{i} & \mathrm{~d} z_{i}
\end{array}\right] \quad(i=0,1,2) .
\end{aligned}
$$

By differentiating ellipsoid surface expression (4.31) and defining $k=1 /\left(1-e^{2}\right)$, one can obtain

$$
\begin{equation*}
\frac{x}{a} \mathrm{~d} x+\frac{y}{a} \mathrm{~d} y+k \frac{z}{a} \mathrm{~d} z=0 \tag{4.67}
\end{equation*}
$$

Combine Equations (4.66) and (4.67) to get

$$
\left[\begin{array}{ccc}
C_{1 x}-C_{0 x} & C_{1 y}-C_{0 y} & C_{1 z}-C_{0 z} \\
C_{2 x}-C_{0 x} & C_{2 y}-C_{0 y} & C_{2 z}-C_{0 z} \\
x / a & y / a & k z / a
\end{array}\right]\left[\begin{array}{c}
\mathrm{d} x \\
\mathrm{~d} y \\
\mathrm{~d} z
\end{array}\right]=\left[\begin{array}{c}
c \mathrm{~d}\left(\Delta t_{1}\right) \\
c \mathrm{~d}\left(\Delta t_{2}\right) \\
0
\end{array}\right]+\left[\begin{array}{c}
\mathbf{u}_{1} \cdot \mathbf{d}_{1}-\mathbf{u}_{0} \cdot \mathbf{d}_{0} \\
\mathbf{u}_{2} \cdot \mathbf{d}_{2}-\mathbf{u}_{0} \cdot \mathbf{d}_{0} \\
0
\end{array}\right]
$$

This can be expressed as

$$
\begin{equation*}
\mathbf{C d} \mathbf{X}=\mathrm{d} \mathbf{Y}=c \mathrm{~d} \mathbf{T}+\mathrm{d} \mathbf{U} \tag{4.68}
\end{equation*}
$$

Assume that the position error of all satellites is uncorrelated with each other and is also uncorrelated with the TDOA measurement, the geolocation error caused by the satellite position random error and the TDOA measurement error can be expressed separately, so the geolocation covariance matrix in ECEF coordinates is

$$
\begin{equation*}
P_{\mathrm{d} \mathbf{x}}=E\left\{\mathrm{~d} \mathbf{X} \mathrm{~d} \mathbf{X}^{\mathrm{T}}\right\}=E\left\{\mathbf{C}^{-1} \mathrm{~d} \mathbf{Y} \mathrm{~d} \mathbf{Y}^{\mathrm{T}} \mathbf{C}^{-\mathrm{T}}\right\}=\mathbf{C}^{-1}\left[c^{2} E\left\{\mathrm{~d} \mathbf{T} \mathrm{~d} \mathbf{T}^{\mathrm{T}}\right\}+E\left\{\mathrm{~d} \mathbf{U} \mathrm{~d} \mathbf{U}^{\mathrm{T}}\right\}\right] \mathbf{C}^{-\mathrm{T}} . \tag{4.69}
\end{equation*}
$$

Expression (4.69) obtains the 3D geolocation covariance matrix of the geolocation error in ECEF coordinates. Considering that accuracy index of the ground emitter location is the horizontal geolocation error, the covariance matrix should be transformed into the top-centric horizon coordinates system of the emitter and its horizontal component should be taken out as the geolocation error:

$$
\begin{equation*}
P_{\mathrm{d} \mathbf{X}}^{\prime}=\left(\mathbf{C}_{s}^{g}\right)_{t} P_{\mathrm{d} \mathbf{X}}\left(\mathbf{C}_{s}^{g}\right)_{t}^{\mathrm{T}}, \tag{4.70}
\end{equation*}
$$

where $\left(\mathbf{C}_{s}^{g}\right)_{t}$ represents the coordinate transformation matrix at the emitter point. The random horizontal geolocation error is

$$
G D O P=\sqrt{P_{\mathrm{d} \mathbf{X}}^{\prime}(1,1)+P_{\mathrm{d} \mathbf{X}}^{\prime}(2,2)}
$$

### 4.4.1.2 Calculation of the Theoretical Geolocation Error Distribution

For a random observation error, the geolocation error caused by the measurement error can be determined separately when the TDOA measurement is uncorrelated with the satellite location measurement. The simulation parameter is listed as below: the satellite altitude is 1000 km and the positions of the satellites, respectively, are $\mathrm{O}_{0}\left(125.21^{\circ} \mathrm{E}, 24.44^{\circ} \mathrm{N}\right), \mathrm{O}_{1}\left(125.68^{\circ} \mathrm{E}\right.$, $\left.25.09^{\circ} \mathrm{N}\right)$, and $\mathrm{O}_{2}\left(125.99^{\circ} \mathrm{E}, 24.44^{\circ} \mathrm{N}\right)$. The radius of the spherical model is the radius vector of the subsatellite geocenter of the earth at the subsatellite of the primary satellite $\mathrm{O}_{0}$. Figure 4.15a shows the geolocation error GDOP (geometric dilution of precision) distribution when the correlation coefficient of the TDOA measurement is 0.5 and the TDOA accuracy $\sigma_{\Delta t}=100 \mathrm{~ns}$. Figure 4.15 b shows the geolocation error GDOP distribution when the satellite position error is $\sigma_{s}=30 \mathrm{~m}$ in each direction of ECEF coordinates. In the figure, ${ }^{*}$ r represents the subsatellite position of the satellites and the number on each curve represents the contour of locating accuracy, with the unit of meter.


Figure 4.15 The GDOP distribution of the random measurement error (meter): (a) $\sigma_{\Delta t}=100 \mathrm{~ns}$ and (b) $\sigma_{s}=30 \mathrm{~m}$

It is shown from Figure 4.15 that the contour of geolocation accuracy caused by the TDOA error and the satellite location error basically is an ellipse centering on the subsatellite point of a three-satellite constellation. Geolocation accuracy near the center of the subsatellite point is the highest. The further the emitter point is away from the center, the larger the geolocation error is and the denser the contour line of the error is, which indicates an increase in the error increment. However, the error increment near the subsatellite point is relatively slow. In the case of normal TDOA accuracy, geolocation accuracy of the three-satellite TDOA method near the subsatellite point can reach about 1 km , which shows that it may be a highly accurate geolocation method.
Under the circumstance of highly accurate satellite navigation (e.g., use of the differential GPS (global positioning system) equipment), the relative location error of satellites can be decreased to decimeter-level and, at such a time, the location error can be omitted. However, when the GPS satellite resource is not available, the satellite location error may be a significant error factor.

### 4.4.2 Analysis of Bias Caused by Altitude Assumption

In Section 4.2, we assumed a zero-altitude to solve the emitter location with two TDOA observations, and further specified the equation of the earth's surface where the emitter is located. The altitude assumption will inevitably lead to bias in geolocation of the emitter. Sensitivity of the three-satellite system to altitude error determines whether the system can be useful in practice.
For the convenience of research, this problem is discussed in the local top-centric horizon coordinates system of the emitter. Refer to Section 2.2 for the transformation relationship between the top-centric horizon coordinates system and the ECEF coordinates. As the emitter altitude is very small compared with the height of the satellite, the locating surface can be approximated with its tangent plant near the emitter point. Suppose the local top-centric horizon coordinates of the satellite are $\mathrm{O}_{0}{ }^{\prime}=\left(x_{0}{ }^{\prime}, y_{0}{ }^{\prime}, z_{0}{ }^{\prime}\right), \mathrm{O}_{1}{ }^{\prime}=\left(x_{1}{ }^{\prime}, y_{1}{ }^{\prime}, z_{1}{ }^{\prime}\right)$, and $\mathrm{O}_{2}{ }^{\prime}=$ $\left(x_{2}{ }^{\prime}, y_{2}{ }^{\prime}, z_{2}{ }^{\prime}\right)$ after the coordinate transformation and the coordinate of the emitter is $\mathrm{T}^{\prime}=$ $\left(x^{\prime}, y^{\prime}, z^{\prime}\right)$. Rewrite the hyperboloid equation as follows:

$$
\begin{align*}
\Delta r_{i}=c \Delta t_{i} & =\sqrt{\left(x^{\prime}-x_{i}^{\prime}\right)^{2}+\left(y^{\prime}-y_{i}^{\prime}\right)^{2}+\left(z^{\prime}-z_{i}^{\prime}\right)^{2}} \\
& -\sqrt{\left(x^{\prime}-x_{0}\right)^{2}+\left(y^{\prime}-y_{0}\right)^{2}+\left(z^{\prime}-z_{0}^{\prime}\right)^{2}} \quad(i=1,2) . \tag{4.71}
\end{align*}
$$

The two TDOA hyperboloids represented by expression (4.71) can be replaced with its tangent plane in the vicinity of the emitter, where the tangent plane at the emitter location $(0,0,0)$ is

$$
\begin{equation*}
\left(\frac{x_{0}^{\prime}}{r_{0}^{\prime}}-\frac{x_{i}^{\prime}}{r_{i}^{\prime}}\right) x^{\prime}+\left(\frac{y_{0}{ }^{\prime}}{r_{0}^{\prime}}-\frac{y_{i}^{\prime}}{r_{i}^{\prime}}\right) y^{\prime}+\left(\frac{z_{0}^{\prime}}{r_{0}^{\prime}}-\frac{z_{i}^{\prime}}{r_{i}^{\prime}}\right) z^{\prime}=0 \quad(i=1,2), \tag{4.72}
\end{equation*}
$$

where

$$
r_{i}^{\prime}=\sqrt{x_{i}^{\prime 2}+y_{i}^{\prime 2}+z_{i}^{\prime 2}} \quad(i=0,1,2)
$$

Table 4.2 Difference of analytical values of the altitude error and the geolocation error

| Location of emitter point | Distance of the emitter <br> away from the geometric <br> center subsatellite point <br> of constellation/km | Theoretic <br> value of <br> geolocation <br> error (m) | Location <br> solution <br> error (m) | Theoretic <br> value <br> error (m) |  |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Longitude $^{\circ}$ | Latitude/ $^{\circ}$ | 0 | 5.2 | 6.9 | 1.7 |
| 126.37 | 25.80 | 212.5 | 299.0 | 301.8 | 2.8 |
| 126.37 | 23.89 | 425.1 | 602.0 | 604.9 | 2.9 |
| 126.37 | 21.98 | 191.5 | 273.0 | 273.4 | 0.4 |
| 128.28 | 25.80 | 383.1 | 548.2 | 549.0 | 0.8 |
| 130.19 | 25.80 |  |  |  |  |

When the altitude assumption error is $\mathrm{d} H$, the locating surface of the zero-altitude earth ellipsoid can be replaced with the tangent plane $z^{\prime}=-\mathrm{d} H$, giving

$$
\left[\begin{array}{l}
\frac{x_{0}{ }^{\prime}}{r_{0}^{\prime}}-\frac{x_{1}{ }^{\prime}}{r_{1}^{\prime}} \frac{y_{0}{ }^{\prime}}{r_{0}^{\prime}}-\frac{y_{1}^{\prime}}{r_{1}^{\prime}}  \tag{4.73}\\
\frac{x_{0}^{\prime}}{r_{0}^{\prime}{ }^{\prime}}-\frac{x_{2}^{\prime}}{r_{2}^{\prime}{ }^{\prime}} \frac{y_{0}^{\prime}}{r_{0}^{\prime}{ }^{\prime}}-\frac{y_{2}^{\prime}}{r_{2}^{\prime}{ }^{\prime}}
\end{array}\right]\left[\begin{array}{l}
x^{\prime} \\
y^{\prime}
\end{array}\right]=\left[\begin{array}{l}
\frac{z_{0}^{\prime}}{r_{0}^{\prime}}-\frac{z_{1}^{\prime}}{r_{1}^{\prime}} \\
\frac{z_{0}^{\prime}}{r_{0}^{\prime}{ }^{\prime}}-\frac{z_{2}^{\prime}}{r_{2}^{\prime}}
\end{array}\right] \mathrm{d} H .
$$

This is simply expressed as

$$
\mathbf{G X}_{g}=\mathbf{H} \mathrm{d} H
$$

According to expression (4.73),

$$
\begin{equation*}
\mathbf{X}_{g}=\mathbf{G}^{-1} \mathbf{H} \mathrm{~d} H \triangleq \mathbf{D} \mathrm{~d} H \tag{4.74}
\end{equation*}
$$

where $\mathbf{D}$ is only determined by the geometric distribution of the three satellites against the emitter. It is revealed from expression (4.74) that the altitude bias can be approximately expressed by a kind of linear relation, where $\mathbf{D}$ represents the GDOP of such a linear relation.
Define the bias index as

$$
\begin{equation*}
\sigma_{s H}=\sqrt{x^{\prime 2}+y^{\prime 2}} \tag{4.75}
\end{equation*}
$$

Under the simulation conditions listed in Section 4.4.1, Table 4.2 shows the difference between the geolocation error from the altitude error analysis and the geolocation error from the geolocation solution in the case of 1 km altitude assumption. Figure 4.16 shows the altitude bias when the altitude of the emitter is 1 km to 200 m without a TDOA measurement error, respectively.
According to Table 4.2 and Figure 4.16, it can be concluded that:

1. The theoretic value of the altitude assumption error is consistent with the geolocation error distribution obtained by a location solution, and the difference is less than several meters, which is very small compared with other geolocation errors and can be omitted. Therefore, the theoretic method of altitude assumption error is effective.
2. For the geolocation error caused by the altitude assumption, the error near the subsatellite point is the smallest and the influence incurred by the altitude assumption error will be more intensive if further away from the subsatellite center. The reason is that the inclination of
the TDOA surface and the earth's surface is approximately $90^{\circ}$ near the subsatellite point; therefore, the altitude error vertical to the earth's surface imposes only a small effect upon the horizontal geolocation error (as listed in Table 4.2). When the emitter is far away from the subsatellite point, the included angle between the TDOA surface where the emitter is located and the earth's surface becomes small, and the influence caused by the altitude error is more obvious.
3. Bias caused by the altitude assumption is basically distributed in an ellipse shape centering on the subsatellite point at the location of the satellite constellation center, and the error contour line is basically distributed in an even way.
4. When the emitter prior altitude assumption bias is about 1 km , the geolocation error caused by the zero-altitude assumption is relatively large, so the geolocation accuracy is not very good. However, as shown in expression (4.74), the geolocation error caused by the altitude assumption is in an approximate linear relation with the altitude bias. Therefore, the altitude bias can be omitted compared with the random geolocation error when the emitter altitude is about $100-200 \mathrm{~m}$. The location algorithm is applicable to locating the emitter adjacent to the geoidal surface.

### 4.4.3 Influence of Change of the Constellation Geometric Configuration on GDOP

The geolocation accuracy distribution under certain satellite geometric configuration conditions were researched in Section 4.4.1. In fact, three satellites may not be located in the same orbit, so the constellation geometric structure is changed continuously when the satellite orbits the earth. It is required to analyze the influence of constellation configuration structure upon geolocation accuracy of the emitter within a coverage area. Figures 4.17 to 4.19 show the distribution of the geolocation error respectively caused by the zero-altitude assumption, the TDOA measurement error, and the satellite position measurement error when the constellation runs at different times (corresponding to different constellation structures). Parameters of the constellation are as follows: the orbital altitude of the primary satellite is 1000 km , eccentricity is 0 (circular orbit), orbit inclination is $60^{\circ}$, RAAN is $120^{\circ}$, argument of perigee is $0^{\circ}$, and time of perigee is 0 second; auxiliary satellites on the same track are 100 km ahead of the primary satellite and the ascending node of the primary satellite not on the same track moves 100 km to the east. The minimum observation angle of the satellite is $25^{\circ}$ and the TDOA measurement error is $\sigma_{\Delta t}=100 \mathrm{~ns}$.
It is revealed from the figures that different structures of the constellation have different geolocation performances to the subsatellite coverage area. Intuitively, when the constellation is located in a low-latitude area and the triangle formed by the ground projection of the three satellites is close to a regular triangle, its geolocation accuracy stays at the highest level; the flatter the projection triangle, the more influence the error imposes and the more the geolocation error is. When the projection of the three satellites is nearly a line in the high latitude area, the geolocation error is very large - sometimes even the no-solution case will happen.


Figure 4.16 Bias distribution in the case of 1 km to 200 m of altitude error: (a) $\mathrm{d} H=1 \mathrm{~km}$ and (b) $\mathrm{d} H=200 \mathrm{~m}$


Figure 4.17 Distribution of the geolocation error caused by a 1 km zero-altitude assumption at different times. Distribution of the error caused by the zero-altitude assumption when (a) $t=0$ second, (b) $t=1000$ seconds, and (c) $t=1530$ seconds

(c)

Figure 4.17 (continued)

### 4.5 Calibration Method of the Three-Satellite TDOA Geolocation System

Besides the altitude assumption error and the TDOA random error, measurement of the TDOA may be influenced by clock drift, delay of a receiver channel, intersatellite clock synchronization error, and so on. Therefore, the measurement of TDOA may suffer bias. Meanwhile, the satellite position (Ephemeris) may also have errors, which may result in bias in the three-satellite TDOA geolocation system. If there are some emitters whose positions are known on the ground, the calibration method can be used to reduce this bias, thereby improving geolocation accuracy. This section will introduce the method to improve accuracy of the geolocating system by calibration.

### 4.5.1 Four-Station Calibration Method and Analysis

The error in the position of the overall satellite constellation should have little influence upon geolocation accuracy, but an error in the relative position of an auxiliary satellite and clock synchronization to the primary satellite may be the main influencing factor. Without considering the location error of the primary satellite, we hope to get an analytical solution of the relative position of auxiliary satellites and the clock synchronization error. Suppose bias of the clock synchronization between auxiliary satellites and the primary satellite are $\delta_{\Delta T_{01}}$ and $\delta_{\Delta T_{02}}$, respectively, and the relative position errors of two auxiliary satellites are $\Delta \mathbf{X}_{1}\left(\Delta x_{1}, \Delta y_{1}, \Delta z_{1}\right)$ and $\Delta \mathbf{X}_{2}\left(\Delta x_{2}, \Delta y_{2}, \Delta z_{2}\right)$, including eight unknown numbers in total. Without considering an intersatellite data link, four ground stations and the corresponding eight TDOA measurement


Figure 4.18 Error GDOP caused by a 100 ns TDOA measurement error at different times. Error GDOP caused by the TDOA measurement error when (a) $t=0$ second, (b) $t=1000$ seconds, and (c) $t=1530$ seconds

(c)

Figure 4.18 (continued)
equations are required to get a solution. If an intersatellite data link exists, that is, the distance between satellites information, there are two relative distance (range) expressions. In this case, only three ground calibration stations and six TDOA equations are required to find a solution. A four-station calibration algorithm is derived as follows.

### 4.5.1.1 Four-Station Calibration Method

Assume there are four calibration stations on the ground, respectively written as $b_{i}(i=1,2$, 3, 4), and suppose its position under ECEF coordinates is $\mathbf{X}_{b_{i}}=\left[\begin{array}{lll}x_{b_{i}} & y_{b_{i}} & z_{b_{i}}\end{array}\right]^{\mathrm{T}}$. Suppose the position of the primary satellite is $\mathbf{X}_{0}=\left[\begin{array}{lll}x_{0} & y_{0} & z_{0}\end{array}\right]^{\mathrm{T}}$ and the positions of auxiliary satellites 1 and 2, respectively, are $\mathbf{X}_{1}=\left[\begin{array}{lll}x_{1} & y_{1} & z_{1}\end{array}\right]^{\mathrm{T}}$ and $\mathbf{X}_{2}=\left[\begin{array}{ll}x_{2} & y_{2} \\ z_{2}\end{array}\right]^{\mathrm{T}}$. Then the TDOA between the primary satellite and the auxiliary satellite 1 is

$$
\begin{equation*}
c \Delta T_{1}^{b_{i}}=\sqrt{\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}}\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)}-\sqrt{\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{1}\right)^{\mathrm{T}}\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{1}\right)} \quad(i=1,2,3,4) \tag{4.76}
\end{equation*}
$$

where $c$ is the propagation speed of the electromagnetic wave.
Errors always exist in actual TDOA measurements with bias and random error. Suppose all random errors are zero-mean and smaller than bias. Therefore, only the influence from the relative position error of auxiliary satellite 1 is taken into account here (as auxiliary satellite 2 is similar to auxiliary satellite 1 , here we will only take auxiliary satellite 1 as the example) and only the relative position error and the TDOA measurement error of auxiliary satellite 1 is considered (as the error of the overall constellation position has little influence on TDOA geolocation accuracy but only causes horizontal movement of the locating plane, its influence


Figure 4.19 GDOP caused by a satellite position measurement error at different times (the satellite position measurement error is 30 m in all directions). Error GDOP caused by the satellite position measurement error when (a) $t=0$ second, (b) $t=1000$ seconds, and (c) $t=1530$ seconds

(c)

Figure 4.19 (continued)
can be omitted during the following derivation). Thus the TDOA of auxiliary satellite 1 that is actually measured is

$$
\begin{equation*}
c \Delta T_{1 m}^{b_{i}}=\sqrt{\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}}\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)}-\sqrt{\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{1}\right)^{\mathrm{T}}\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{1}\right)}+c \delta_{\Delta T 1 s} \quad(i=1,2,3,4) \tag{4.77}
\end{equation*}
$$

The bias in the relative position of auxiliary satellite 1 is aliased into the relative position of auxiliary satellite 1 . Therefore, only if the relative position $\mathbf{X}_{d 1 s}$ of auxiliary satellite 1 is estimated accurately can the bias of the auxiliary satellite's position be eliminated. Now, the problem is how to estimate $\delta_{\Delta T 1 s}$ and $\mathbf{X}_{d 1 s}$ based on $\Delta T_{1 m}^{b_{i}}(i=1,2,3,4)$. Since there are four unknowns, four calibration stations are required to fully solve $\delta_{\Delta T 1 s}$ and $\mathbf{X}_{d 1 s}$.
Transposing expression (4.77) yields

$$
\begin{equation*}
\sqrt{\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{1}\right)^{\mathrm{T}}\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{1}\right)}=\sqrt{\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}}\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)}-c \Delta T_{1 m}^{b_{i}}+c \delta_{\Delta T 1 s} \quad(i=1,2,3,4) \tag{4.78}
\end{equation*}
$$

where

$$
\begin{aligned}
& \sqrt{\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{1}\right)^{\mathrm{T}}\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{1}\right)} \\
& \quad=\sqrt{\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}+\mathbf{X}_{d 1 s}\right)^{\mathrm{T}}\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}+\mathbf{X}_{d 1 s}\right)} \\
& \quad=\sqrt{\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}}\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)+2\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}} \mathbf{X}_{d 1 s}+\mathbf{X}_{d 1 s}^{\mathrm{T}} \mathbf{X}_{d 1 s}}
\end{aligned}
$$

Square expression (4.78) and let $r_{0}^{b_{i}}=\sqrt{\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}}\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)}$. Simplifying it yields

$$
\begin{align*}
2\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}} \mathbf{X}_{d 1 s}+\mathbf{X}_{d 1 s}^{\mathrm{T}} \mathbf{X}_{d 1 s}= & \left(c \Delta T_{1 m}^{b_{i}}\right)^{2}+\left(c \delta_{\Delta T 1 s}\right)^{2} \\
& -2 r_{0}^{b_{i}} c \Delta T_{1 m}^{b_{i}}+2\left(r_{0}^{b_{i}}-\Delta T_{1 m}^{b_{i}}\right) c \delta_{\Delta T 1 s} \quad(i=1,2,3,4) \tag{4.79}
\end{align*}
$$

Subtract the equation in which $i=2,3,4$ and the one in which $i=1$ of expression (4.79). Then

$$
\begin{align*}
2\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{b_{1}}\right)^{\mathrm{T}} \mathbf{X}_{d 1 s}= & \left(c \Delta T_{1 m}^{b_{i}}\right)^{2}-\left(c \Delta T_{1 m}^{b_{1}}\right)^{2}-2 r_{0}^{b_{i}} c \Delta T_{1 m}^{b_{i}}+2 r_{0}^{b_{1}} c \Delta T_{1 m}^{b_{1}} \\
& +2\left(r_{0}^{b_{i}}-\Delta T_{1 m}^{b_{i}}-r_{0}^{b_{1}}+\Delta T_{1 m}^{b_{1}}\right) c \delta_{\Delta T 1 s} \quad(i=2,3,4) \tag{4.80}
\end{align*}
$$

Let

$$
\begin{aligned}
k_{1}^{b_{i}}= & 2 c\left(r_{0}^{b_{i}}-\Delta T_{1 m}^{b_{i}}-r_{0}^{b_{1}}+\Delta T_{1 m}^{b_{1}}\right), k_{2}^{b_{i}}=\left(c \Delta T_{1 m}^{b_{i}}\right)^{2}-\left(c \Delta T_{1 m}^{b_{1}}\right)^{2} \\
& -2 r_{0}^{b_{i}} c \Delta T_{1 m}^{b_{i}}+2 r_{0}^{b_{1}} c \Delta T_{1 m}^{b_{1}} .
\end{aligned}
$$

Expression (4.80) can be expressed as

$$
\begin{equation*}
2\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{b_{1}}\right)^{\mathrm{T}} \mathbf{X}_{d 1 s}=k_{1}^{b_{i}} \delta_{\Delta T 1 s}+k_{2}^{b_{i}} \quad(i=2,3,4) \tag{4.81}
\end{equation*}
$$

Let

$$
\left.\left.\begin{array}{rl}
\mathbf{A} & =\left[\left(\mathbf{X}_{b_{2}}-\mathbf{X}_{b_{1}}\right)\left(\mathbf{X}_{b_{3}}-\mathbf{X}_{b_{1}}\right)\left(\mathbf{X}_{b_{4}}-\mathbf{X}_{b_{1}}\right)\right.
\end{array}\right]^{\mathrm{T}}, ~ 子 \begin{array}{lll}
k_{2}^{b_{2}} & k_{2}^{b_{3}} & k_{2}^{b_{4}}
\end{array}\right)^{\mathrm{T}} .
$$

One can obtain

$$
\begin{equation*}
\mathbf{X}_{d 1 s}=\frac{1}{2} \mathbf{A}^{-1} \mathbf{K}_{1} \delta_{\Delta T 1 s}+\frac{1}{2} \mathbf{A}^{-1} \mathbf{K}_{2}=\mathbf{K}_{3} \delta_{\Delta T 1 s}+\mathbf{K}_{4} . \tag{4.82}
\end{equation*}
$$

Substitute expression (4.82) of $\mathbf{X}_{d 1 s}$ into expression (4.79), where

$$
\begin{aligned}
\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}} \mathbf{X}_{d 1 s} & =\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}}\left(\mathbf{K}_{3} \delta_{\Delta T 1 s}+\mathbf{K}_{4}\right) \\
& =\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}} \mathbf{K}_{3} \delta_{\Delta T 1 s}+\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}} \mathbf{K}_{4}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{X}_{d 1 s}^{\mathrm{T}} \mathbf{X}_{d 1 s} & =\left(\mathbf{K}_{3} \delta_{\Delta T 1 s}+\mathbf{K}_{4}\right)^{\mathrm{T}}\left(\mathbf{K}_{3} \delta_{\Delta T 1 s}+\mathbf{K}_{4}\right) \\
& =\mathbf{K}_{3}^{\mathrm{T}} \mathbf{K}_{3} \delta_{\Delta T 1 s}^{2}+2 \mathbf{K}_{3}^{\mathrm{T}} \mathbf{K}_{4} \delta_{\Delta T 1 s}+\mathbf{K}_{4}^{\mathrm{T}} \mathbf{K}_{4} .
\end{aligned}
$$

After $i=1$ is substituted, expression (4.79) can be written as

$$
\begin{equation*}
a_{1} \delta_{\Delta T 1 s}^{2}+b_{1} \delta_{\Delta T 1 s}+c_{1}=0 \tag{4.83}
\end{equation*}
$$

where $a_{1}=\mathbf{K}_{3}^{\mathrm{T}} \mathbf{K}_{3}-c^{2}, b_{1}=2 \mathbf{K}_{3}^{\mathrm{T}} \mathbf{K}_{4}+2\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}} \mathbf{K}_{3}-2\left(r_{0}^{b_{i}}-c \Delta T_{1 m}^{b_{i}}\right)$, and $c_{1}=\mathbf{K}_{4}^{\mathrm{T}} \mathbf{K}_{4}+$ $2\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}} \mathbf{K}_{4}-\left(c \Delta T_{1 m}^{b_{i}}\right)^{2}+2 r_{0}^{b_{i}} c \Delta T_{1 m}^{b_{i}}$.

One can solve the quadratic equation and find roots, from Equation (4.83):

$$
\begin{equation*}
\delta_{\Delta T 1 s}=\frac{-b_{1} \pm \sqrt{b_{1}^{2}-4 a_{1} c_{1}}}{2 a_{1}} \tag{4.84}
\end{equation*}
$$

Substitute this into expression (4.82) to get

$$
\begin{equation*}
\mathbf{X}_{d 1 s}=\mathbf{K}_{3} \delta_{\Delta T 1 s}+\mathbf{K}_{4} . \tag{4.85}
\end{equation*}
$$

The expression above has two groups of solutions. As one group of solutions is inconsistent with the physical circumstances it should be deleted. We can then obtain the analytical solution of the clock synchronization error and the relative position of the auxiliary satellite in the three-satellite geolocation system. Use the solution of the clock synchronization error and the relative position to compensate for TDOA measurement bias and use the satellite positions to solve the position of the emitter, thereby reducing the influence of TDOA bias and improving geolocation accuracy.

### 4.5.1.2 Four-Station Calibration Geometric Configuration

The principle of four-station calibration is the same as that of four-station TDOA location, which is similar to the inverse of ground multistation TDOA location. Therefore, we can employ research achievements about multistation TDOA location to research the geometric configuration problem of multistation calibration. According to research achievements about geometric configurations in references [8] and [13], the inverted triangle-shape station geometric configuration and the ' Y '-shape (also called star-shape) station geometric configuration enjoy the highest geolocation accuracy among all geometric configuration modes. Compared with the ' Y '-shape (star-shape) station geometric configuration, the inverted triangle-shape configuration is more accurate in geolocation accuracy. However, such a configuration is difficult to realize in practice and its geolocation accuracy is obvious in anisotropic directions (the geolocation accuracy stays relatively poor in some directions). The longer the length of the baseline between stations, the higher is the geolocation accuracy.
It is revealed from expressions (4.81) and (4.82) that expression (4.82) has solutions when the four calibration stations are not within the same plane, that is, $\operatorname{rank}(\mathbf{A})=3$. Considering the actual station geometric configuration status, three calibration stations can be arranged in one plane and the fourth calibration station may be arranged on another plane. The longer the length of the baseline, the higher is the geolocation accuracy; however, an appropriate baseline length can be selected due to limitations of various factors, such as the satellite instantaneous coverage area and geographical conditions. Refer to Figure 4.20 for two typical station geometric configurations.
Considering various factors, the ' Y '-shape (star-shape) station geometric configuration is more suitable for the geometric configuration of the three-satellite geolocation system with ground four-station calibration.


Figure 4.20 Two typical station geometric configurations. (a) Inverted triangle-shape station geometric configuration and (b) ' Y '-shape station geometric configuration

### 4.5.1.3 Simulation

In this section, computer simulation is used to verify the accuracy of the four-station calibration algorithm and its improvement effects upon the system's geolocation accuracy.
The simulation conditions are as follows: the actual location data of the satellites is generated by the STK (satellite tool kit); the location of the four ground stations are respectively
set as $\left(102.7064^{\circ} \mathrm{E}, 25.0366^{\circ} \mathrm{N}\right),\left(109.4927^{\circ} \mathrm{E}, 18.2588^{\circ} \mathrm{N}\right),\left(109.3735^{\circ} \mathrm{E}, 24.3152^{\circ} \mathrm{N}\right)$, and $\left(112.9685^{\circ} \mathrm{E}, 28.1976^{\circ} \mathrm{N}\right)$ to form an approximate ' Y '-shape geometric configuration; the location of the three satellites is $\mathrm{O}_{0}\left(122.0357^{\circ} \mathrm{E}, 36.4243^{\circ} \mathrm{N}\right), \mathrm{O}_{1}\left(120.9255^{\circ} \mathrm{E}, 36.3627^{\circ} \mathrm{N}\right)$, and $\mathrm{O}_{2}\left(121.6992^{\circ} \mathrm{E}, 37.2870^{\circ} \mathrm{N}\right)$; the bias of the TDOA measurement is 60 ns ; the RMS of the random error is 20 ns (normal distribution); the overall location error of the constellations is 150 m ; the relative position error of the satellites is 50 m ; ; $\mathrm{O}^{\prime}$ in simulation results represents the location of calibration stations; and '*' represents the position of the satellites' subsatellite point. Carry out the Monte Carlo tests 100 times to evaluate the geolocation error. We can obtain simulation results of the geolocation error as shown in Figure 4.21. In this figure, the error unit is the meter, and the $X$ axis and $Y$ axis, respectively, serve as the longitude and latitude.
It is revealed from Figure 4.21 that the contour map of the accuracy distribution is changed before and after calibration and that the accuracy distribution contour after calibration is more like a circle. Geolocation error within $1000 \mathrm{~km}^{2}$ of the vicinity of the subsatellite point is decreased to within 200 and 500 m after calibration from 500 to 1000 m before calibration. This represents a double increase in accuracy. However, if the emitter is 2000 km away from the subsatellite point, the geolocation accuracy is not obviously improved.

### 4.5.2 Three-Station Calibration Method

If there is ranging equipment on the satellites, the distance between satellite information can be measured. Common ranging ways including transponder and laser ranging.
Transponder ranging means that an enquiring signal is transmitted by the primary satellite to an auxiliary satellite, which will transmit an answering signal upon receipt of the enquiring signal. A time mark is attached in the answering signal based on its own clock. After receiving the acknowledgment signal, the primary satellite will calculate the transmission time of the signal among satellites to obtain the distances between the primary satellite and the auxiliary satellites.
The laser ranging mode involves a laser transmitter carried by the primary satellite and a corner reflector by the auxiliary satellite. The distance between the satellites can be obtained by measuring the time delay value of the laser pulse reflection.
A three-station calibration algorithm employing the two different calibration algorithms is derived as follows.

### 4.5.2.1 Three-Station Calibration Method

If intersatellite range information exists, the relative position error and the clock synchronization error of the auxiliary satellite can be solved via only three ground calibration stations. Two different calibration methods of three-station location to estimate the bias are listed below:

## 1. Method I

Assume that there is ranging information between the primary satellite and auxiliary satellite 1 . Let the distance be

$$
\begin{equation*}
r_{1}=\sqrt{\mathbf{X}_{d 1 s}^{\mathrm{T}} \mathbf{X}_{d 1 s}} \tag{4.86}
\end{equation*}
$$

where $\mathbf{X}_{d 1 s}$ is the distance vector between the primary satellite and auxiliary satellite.


Figure 4.21 GDOP of geolocation for the four-station calibration algorithm. GDOP contour of the geolocation error (a) before calibration and (b) after calibration

When transponder ranging is used, due to bias of intersatellite clock synchronization, the measured distance between the primary satellite and auxiliary satellite 1 is

$$
\begin{equation*}
r_{1}=\sqrt{\mathbf{X}_{d 1 s}^{\mathrm{T}} \mathbf{X}_{d 1 s}}+c \delta_{\Delta T 1 s} \tag{4.87}
\end{equation*}
$$

Then we can get

$$
\begin{equation*}
\mathbf{X}_{d 1 s}^{\mathrm{T}} \mathbf{X}_{d 1 s}=r_{1 m}^{2}+\left(c \delta_{\Delta T 1 s}\right)^{2}-2 r_{1 m} c \delta_{\Delta T 1 s} \tag{4.88}
\end{equation*}
$$

Substituting this into expression (4.79) gives

$$
\begin{align*}
2\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}} \mathbf{X}_{d 1 s}= & 2 r_{1 m} c \delta_{\Delta T 1 s}-r_{1 m}^{2}+\left(c \Delta T_{1 m}^{b_{i}}\right)^{2}+2 r_{0}^{b_{i}} c \Delta T_{1 m}^{b_{i}} \\
& +2\left(r_{0}^{b_{i}}-\Delta T_{1 m}^{b_{i}}\right) c \delta_{\Delta T 1 s} \quad(i=1,2,3) . \tag{4.89}
\end{align*}
$$

Then let

$$
k_{1}^{b_{i}}=2\left(r_{0}^{b_{i}}-c \Delta T_{1 m}^{b_{i}}\right)+2 r_{1 m} c, k_{2}^{b_{i}}=\left(c \Delta T_{1 m}^{b_{i}}\right)^{2}-2 r_{0}^{b_{i}} c \Delta T_{1 m}^{b_{i}}-r_{1 m}^{2} \quad(i=1,2,3)
$$

Expression (4.89) can therefore be expressed as

$$
\begin{equation*}
2\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}} \mathbf{X}_{d 1 s}=k_{1}^{b_{i}} \delta_{\Delta T 1 s}+k_{2}^{b_{i}} \quad(i=1,2,3) \tag{4.90}
\end{equation*}
$$

Let $\quad \mathbf{A}=\left[\left(\mathbf{X}_{b_{2}}-\mathbf{X}_{b_{1}}\right)\left(\mathbf{X}_{b_{3}}-\mathbf{X}_{b_{1}}\right)\left(\mathbf{X}_{b_{4}}-\mathbf{X}_{b_{1}}\right)\right]^{\mathrm{T}}, \quad \mathbf{K}_{1}=\left(\begin{array}{ll}k_{1}^{b_{2}} & k_{1}^{b_{3}} \\ k_{1}^{b_{4}}\end{array}\right)^{\mathrm{T}}, \quad$ and $\mathbf{K}_{2}=\left(k_{2}^{b_{2}} k_{2}^{b_{3}} k_{2}^{b_{4}}\right)^{\mathrm{T}}$, which yields

$$
\begin{equation*}
\mathbf{X}_{d 1 s}=\frac{1}{2} \mathbf{A}^{-1} \mathbf{K}_{1} \delta_{\Delta T 1 s}+\frac{1}{2} \mathbf{A}^{-1} \mathbf{K}_{2}=\mathbf{K}_{3} \delta_{\Delta T 1 s}+\mathbf{K}_{4} . \tag{4.91}
\end{equation*}
$$

Therefore

$$
\mathbf{X}_{d 1 s}^{\mathrm{T}} \mathbf{X}_{d 1 s}=\left(\mathbf{K}_{3} \delta_{\Delta T 1 s}+\mathbf{K}_{4}\right)^{\mathrm{T}}\left(\mathbf{K}_{3} \delta_{\Delta T 1 s}+\mathbf{K}_{4}\right)=\mathbf{K}_{3}^{\mathrm{T}} \mathbf{K}_{3} \delta_{\Delta T 1 s}^{2}+2 \mathbf{K}_{3}^{\mathrm{T}} \mathbf{K}_{4} \delta_{\Delta T 1 s}+\mathbf{K}_{4}^{\mathrm{T}} \mathbf{K}_{4} .
$$

Substituting this into expression (4.88) gives

$$
\begin{equation*}
a_{1} \delta_{\Delta T 1 s}^{2}+b_{1} \delta_{\Delta T 1 s}+c_{1}=0, \tag{4.92}
\end{equation*}
$$

where $a_{1}=\mathbf{K}_{3}^{\mathrm{T}} \mathbf{K}_{3}-c^{2}, b_{1}=2 \mathbf{K}_{3}^{\mathrm{T}} \mathbf{K}_{4}-2 r_{1 m} c$, and $c_{1}=\mathbf{K}_{4}^{\mathrm{T}} \mathbf{K}_{4}-r_{1 m}^{2}$.
The expression above is a quadratic equation. Solve it and find the roots:

$$
\begin{equation*}
\delta_{\Delta T 1 s}=\frac{-b_{1} \pm \sqrt{b_{1}^{2}-4 a_{1} c_{1}}}{2 a_{1}} \tag{4.93}
\end{equation*}
$$

Substituting this into expression (4.91) gives

$$
\begin{equation*}
\mathbf{X}_{d 1 s}=\mathbf{K}_{3} \delta_{\Delta T 1 s}+\mathbf{K}_{4} . \tag{4.94}
\end{equation*}
$$

This expression has two groups of solutions. As one group of solution in inconsistent with physical circumstances should be deleted. We can obtain the analytical solution of the clock synchronization error of the three-satellite geolocation system and the relative position of the auxiliary satellite.
2. Method II

Suppose there is intersatellite distance information between the primary satellite and auxiliary satellite 1 . Let the distance be

$$
\begin{equation*}
r_{1}=\sqrt{\mathbf{X}_{d 1 s}^{\mathrm{T}} \mathbf{X}_{d 1 s}} \tag{4.95}
\end{equation*}
$$

When the laser ranging mode is used, the ranging information is accurate and does not contain clock synchronization bias or error. One can obtain

$$
\begin{align*}
2\left(\mathbf{X}_{b_{i}}-\mathbf{X}_{0}\right)^{\mathrm{T}} \mathbf{X}_{d 1 s}= & \left(c \Delta T_{1 m}^{b_{i}}\right)^{2}-r_{1 m}^{2}+\left(c \delta_{\Delta T 1 s}\right)^{2}-2 r_{0}^{b_{i}} c \Delta T_{1 m}^{b_{i}} \\
& +2\left(r_{0}^{b_{i}}-\Delta T_{1 m}^{b_{i}}\right) c \delta_{\Delta T 1 s} \quad(i=1,2,3) \tag{4.96}
\end{align*}
$$

So let $k_{1}=c^{2}, k_{2}^{b_{i}}=2\left(r_{0}^{b_{i}}-c \Delta T_{1 m}^{b_{i}}\right)$, and $k_{3}^{b_{i}}=\left(c \Delta T_{1 m}^{b_{i}}\right)^{2}-2 r_{0}^{b_{i}} c \Delta T_{1 m}^{b_{i}}-r_{1 m}^{2}$. This can be written in the following matrix form:

$$
\begin{equation*}
2 \mathbf{A} \mathbf{X}_{d 1 s}=\mathbf{K}_{1}\left(\delta_{\Delta T 1 s}\right)^{2}+\mathbf{K}_{2} \delta_{\Delta T 1 s}+\mathbf{K}_{3} \tag{4.97}
\end{equation*}
$$

where $\mathbf{A}=\left[\left(\mathbf{X}_{b_{2}}-\mathbf{X}_{b_{1}}\right)\left(\mathbf{X}_{b_{3}}-\mathbf{X}_{b_{1}}\right)\left(\mathbf{X}_{b_{4}}-\mathbf{X}_{b_{1}}\right)\right]^{\mathrm{T}}, \mathbf{K}_{1}=\left(k_{1}^{b_{2}} k_{1}^{b_{3}} k_{1}^{b_{4}}\right)^{\mathrm{T}}$, and $\mathbf{K}_{2}=$ $\left(k_{2}^{b_{2}} k_{2}^{b_{3}} k_{2}^{b_{4}}\right)^{\mathrm{T}}$. Then

$$
\begin{equation*}
\mathbf{X}_{d 1 s}=\frac{1}{2} \mathbf{A}^{-1} \mathbf{K}_{1}\left(\delta_{\Delta T 1 s}\right)^{2}+\frac{1}{2} \mathbf{A}^{-1} \mathbf{K}_{2} \delta_{\Delta T 1 s}+\frac{1}{2} \mathbf{A}^{-1} \mathbf{K}_{3}=\mathbf{K}_{4}\left(\delta_{\Delta T 1 s}\right)^{2}+\mathbf{K}_{5} \delta_{\Delta T 1 s}+\mathbf{K}_{6} \tag{4.98}
\end{equation*}
$$

Substitute expression (4.88) and square it, to obtain

$$
a \delta_{\Delta T 1 s}^{4}+b \delta_{\Delta T 1 s}^{3}+c \delta_{\Delta T 1 s}^{2}+d \delta_{\Delta T 1 s}+e=0
$$

where $a=\mathbf{K}_{4}^{\mathrm{T}} \mathbf{K}_{4}, b=2 \mathbf{K}_{4}^{\mathrm{T}} \mathbf{K}_{5}, c=\mathbf{K}_{5}^{\mathrm{T}} \mathbf{K}_{5}+2 \mathbf{K}_{4}^{\mathrm{T}} \mathbf{K}_{6}, d=\mathbf{K}_{5}^{\mathrm{T}} \mathbf{K}_{6}$, and $e=\mathbf{K}_{6}^{\mathrm{T}} \mathbf{K}_{6}-r_{1}^{2}$. The expression above is a quartic equation. Therefore, the analytical solution method of a quartic equation can be used to solve the clock synchronization error of a three-satellite geolocation system and the relative position of the auxiliary satellite.

### 4.5.2.2 Simulation

In this section, computer simulation is used to verify the accuracy of the three-station calibration algorithm and its improvement effects upon the system's geolocation accuracy. A large number of simulations show that the calibration effects of two ranging modes are the same. Therefore, only simulation results of the laser ranging mode are taken as an example for analysis in this section.


Figure 4.22 Comparison between GDOP contours of the geolocation error before and after three-station calibration. GDOP contour of the geolocation error curve (a) before calibration and (b) after calibration

Use the location scenario similar to that described in the section above. The location of the three ground stations are respectively set as $\left(102.7064^{\circ} \mathrm{E}, 25.0366^{\circ} \mathrm{N}\right),\left(109.4927^{\circ} \mathrm{E}\right.$, $\left.18.2588^{\circ} \mathrm{N}\right)$, and $\left(112.9685^{\circ} \mathrm{E}, 28.1976^{\circ} \mathrm{N}\right)$ to form an approximate equilateral triangle-shape geometric configuration; the location of the three satellites is $\mathrm{O}_{0}\left(108.7248^{\circ} \mathrm{E}, 19.1027^{\circ} \mathrm{N}\right)$, $\mathrm{O}_{1}\left(107.6106^{\circ} \mathrm{E}, 19.0373^{\circ} \mathrm{N}\right)$, and $\mathrm{O}_{2}\left(108.1702^{\circ} \mathrm{E}, 20.0204^{\circ} \mathrm{N}\right)$; the bias of the TDOA measurement is 60 ns ; the RMS of the random error is 20 ns (normal distribution); the absolution location error of the constellations is 150 m ; the relative position error of the satellites is 50 m ; the intersatellite measurement error is 0.3 m ; and the other conditions are the same as those specified in the section above. In this way, the simulation result shown in Figure 4.22 can be obtained. ' O ' in the simulation results represents the location of the calibration stations, and '*' represents the position of the satellites' subsatellite point. In this figure, the error unit is the meter and the $X$ axis and $Y$ axis, respectively, serve as the longitude and latitude.
By comparing the simulation results before and after calibration, Figure 4.22 shows that the geolocation error within $1500 \mathrm{~km}^{2}$ in the vicinity of a subsatellite point is decreased to within 200 and 500 m after calibration from 500, 1000, and 2000 m before calibration, indicating an increase of one to two times. However, for the emitters 2500 km away from the subsatellite point, the geolocation accuracy is not obviously improved.
By comparing the three-station calibration results shown in Figure 4.22b and the four-station calibration results in Figure 4.21b, it is shown that the effect of three-station calibration is better than that of four-station calibration, which is attributed to accurate ranging formation in three-station calibration.

## References

1. Zhou, Y., An, W., Guo, F., et al. (2009) Principles of Electronic Warfare. Beijing: Publishing House of Electronics Industry (in Chinese).
2. Hu, L. (2004) Passive Location. Beijing: National Defence Industry Press (in Chinese).
3. Wang, H., Chen, L., and Ren, X. (2001) Analysis and design of three-satellite TDOA geolocation satellites cluster. Journal of National University of Defense Technology, 23(4): 629 (in Chinese).
4. Wang, H., Ren, X., and Cheng, L. (2000) Localization accuracy analysis of three-satellite TDOA geolocation constellation. Chinese Space Science and Technology, 5: 24-29 (in Chinese).
5. Foy, W.H. (1976) Position-location solutions by Taylor-series estimation. IEEE Transactions on Aerospace and Electronic Systems, AES-12 (2), 187-194.
6. Torrieri, D.J. (1984) Statistical theory of passive location systems. IEEE Transactions on Aerospace and Electronic Systems, AES-20 (2), 183-198.
7. Ho, K. C. and Chan, Y. T. (1997) Geolocation of a know altitude object from TDOA and FDOA measurements, IEEE Transactions on Aerospace and Electronic Systems, 33(3),: 770-783.
8. Zhong, D. (2002) Research on Three-Satellite TDOA Geolocation Based on WGS-84. Changsha: Graduate School of National University of Defense Technology (in Chinese).
9. Zhong, D., Deng, X., and Zhou, Y. (2003) A location method based on WGS-84 earth model using satellites TDOA measurements. Journal of Astronautics, 24(6), November: 569-573 (in Chinese).
10. Xi, X., Wang, W., and Gao, Y. (2003) Fundamentals of Near-Earth Spacecraft Orbit. Changsha: NUDT Publish House, (in Chinese).
11. Poisel, R.A. (2012) Electronic Warfare Target Location Methods, 2 nd edn, Artech House.
12. Sun, Z., Zhou, Y., and He, L. (1996) Active and Passive Localization Technology with Single or Multiple Base. Beijing: National Defence Industry Press (in Chinese).
13. Gao, Q., Guo, F., Wu, J., et al. (2007) Research on a calibration algorithm for the three-satellite TDOA geolocation system. Aerospace Electronic Warfare, 5(104): 5-7 (in Chinese).

## 5

## Dual-Satellite Geolocation Based on TDOA and FDOA

The dual-satellite TDOA-FDOA (time difference of arrival-frequency difference of arrival) geolocation method uses two geometrically separated satellites to passively intercept the signal from the noncooperative emitter on the earth and then calculate the location of the emitter according to the TDOA and FDOA of the signal from the emitter to the satellite. Because both TDOA and FDOA are nonlinear functions of the emitter position in 3D space, the position of the emitter can be determined according to prior information of the emitter on the earth's surface. In the research on dual-satellite TDOA-FDOA geolocation, much concern has been given on how to calculate the location of the emitter, whose parameters have a greater influence on geolocation accuracy, and to what extent the parameter measurements should be accurate. Therefore, geolocation accuracy under different parameter measurement error levels will be analyzed.
Therefore, this chapter first introduces the principle of dual-satellite TDOA-FDOA geolocation and then discusses the solution of the dual LEO (low earth orbit) satellite geolocation expression with two satellites running in and then not in the same orbit, respectively. Finally a geolocation method of TDOA-FDOA geolocation by two HEO (high earth orbit) satellites based on reference stations is introduced, the geolocation accuracy is analyzed, and the technology for measuring TDOA-FDOA from the signal waveform is described.

### 5.1 Introduction of TDOA-FDOA Geolocation by a Dual-Satellite

### 5.1.1 Explanation of Dual-Satellite Geolocation Theory

In the dual-satellite geolocation system, each satellite intercepts the radio signals from the transmitters (or emitters, jamming sources) on earth. TDOA and FDOA can be measured via correlation of the intercepted signals from two satellites. Based on TDOA and FDOA, the position information of the emitter may be obtained. For example, an equal-TDOA hyperboloid can be determined via TDOA measurement and an equal-FDOA surface determined via

[^1]

Figure 5.1 Sketch of the dual-satellite TDOA-FDOA combined geolocation principle

FDOA measurement. For the emitter on land or the emitter on the ocean surface at a known altitude, if the TDOA and FDOA are measured simultaneously, a circle taking the line of two satellites as the axis can be formed by intersecting the equal-TDOA revolution hyperboloid by the equal-FDOA revolution surface. Two points can be obtained by intersecting this circle by the earth's surface if the ambiguous geolocation point can be eliminated and the other point can be the actual location of the emitter, as shown in Figure 5.1 [1-4].
Compared with the single-satellite LOS (line of sight) geolocation method and the three-satellite TDOA geolocation method, the dual-satellite TDOA-FDOA geolocation method is featured with no requirement for a satellite platform attitude measurement and is not affected by a larger geolocation error. The three-satellite TDOA geolocation system requires three satellites at least, so it may be expensive in system realization. In addition, there are always some satellite configurations that cannot perform the geolocation, sometimes because these satellites are not in the same orbit (for example, in the high latitude area). In the dual-satellite TDOA-FDOA geolocation system, two satellites can be configured in the same orbit so as to ensure the consistency of global geolocation accuracy.
For the LEO dual-satellite TDOA-FDOA geolocation system, if the baseline between two satellites is long enough, geolocation accuracy can reach a high level due to the fast moving velocity of the LEO satellite relative to the earth and the great Doppler frequency difference generated therefrom. Compared with other multiplatform geolocations, the TDOA-FDOA geolocation mode reduces the number of platforms and decreases the difficulty of system realization and launching cost, and compared with the single-platform geolocation mode, its geolocation accuracy is relatively high.

### 5.1.2 Structure of Dual-Satellite TDOA-FDOA Geolocation System

As shown in Figure 5.2, the dual-satellite TDOA-FDOA geolocation system uses two satellites to passively intercept signals from the noncooperative emitter on the earth and then estimate the location of the emitter according to the TDOA and FDOA of the signal from the emitter to the satellite with the known satellite ephemeris and satellite moving velocity and by


Figure 5.2 Sketch of the dual-satellite geolocation system


Figure 5.3 Sketch of dual-satellite geolocation processing
supposing the emitter to be on the earth's surface. The system also has a ground geolocation station, used by the user for processing and analysis of the data received by the satellites so as to monitor their operating states, as shown in Figure 5.2.
Generally, one of the two satellites is the primary satellite (satellite A) and the other is the auxiliary satellite (satellite B), as shown in Figure 5.2. The primary satellite receives the main lobe signal of the emitter and the auxiliary satellite receives the side lobe signal (or main lobe signal in certain cases) of the emitter. The two satellites may run in or not in the same orbit. A sketch of the dual-satellite geolocation system is shown in Figure 5.3.
During geolocation, the receivers of the two satellites receive the transmitted signal of an emitter, implement analog-to-digital conversion (A/D), carry out signal processing and FDOA and TDOA parameter estimation, and finally calculate the geolocation results with a satellite orbit parameter (obtained and known by a ground TTC\&M (telemetry, tracking, command, and monitoring) system). Signal processing and data processing work can be completed on
the satellite on a real-time basis or on the ground to post-processing after being sent to the ground station through a satellite-to-ground data link or relay satellite.

### 5.2 Dual LEO Satellite TDOA-FDOA Geolocation Method

To localize an emitter, the double dual LEO satellite geolocation system utilizes the measured TDOA-FDOA parameters of the emitter signals in combination with the restriction condition that the emitter is on the earth's surface and then finds out the solution to the emitter's location. There are two methods for the dual-satellite TDOA-FDOA geolocation: one is to obtain the emitter location by directly solving multivariant nonlinear equations using the analytics method and the other is to obtain the emitter location by solving multivariant nonlinear equations using the iterative method. The analytic method is relatively better for the initial location of the emitter required by the iterative method and its calculation is complex. If the analytic method is adopted to solve the TDOA-FDOA geolocation equation, two cases also exist, that is, two satellites running in or not in the same orbit. In the case of two satellites running in the same orbit, an approximation algorithm is used to simplify the process of finding the solution.

### 5.2.1 Geolocation Model

In this section, the ECEF (earth-center earth-fixed) coordinates system and the geolocation model will be established. In the dual-satellite geolocation system, two satellites and the emitter are as shown in Figure 5.4.
In Figure 5.4, the emitter is located at E , with its position given as $\mathbf{u}=[x, y, z]^{\mathrm{T}}$. The coordinates of two satellites $\mathrm{S}_{1}$ and $\mathrm{S}_{2}$ are defined as $\mathbf{s}_{1}=\left[x_{1}, y_{1}, z_{1}\right]^{\mathrm{T}}$ and $\mathbf{s}_{2}=\left[x_{2}, y_{2}, z_{2}\right]^{\mathrm{T}}$, and their velocities as $\mathbf{v}_{1}=\left[v_{x 1}, v_{y 1}, v_{z 1}\right]^{\mathrm{T}}$ and $\mathbf{v}_{2}=\left[v_{x 2}, v_{y 2}, v_{z 2}\right]^{\mathrm{T}}$, respectively.
The two satellites receive a signal transmitted from the emitter on earth to obtain the measured value of TDOA and FDOA. Define the TDOA of the electromagnetic wave from the emitter to satellite 1 and to satellite 2 as $\Delta t$ and the corresponding FDOA as $\Delta f_{d}$.
Supposing the distance from satellite $i$ to the emitter is $r_{i}$, then

$$
\begin{equation*}
r_{i}=\left(\left(\mathbf{s}_{i}-\mathbf{u}\right)^{\mathrm{T}}\left(\mathbf{s}_{i}-\mathbf{u}\right)\right)^{1 / 2} \quad(i=1,2) \tag{5.1}
\end{equation*}
$$

The TDOA measurement can be seen as the distance difference of the emitter to two satellites. Because the propagation speed of the electromagnetic wave is constant, the measurement expression can be obtained as

$$
\begin{equation*}
\Delta r=c \Delta t=r_{1}-r_{2}, \tag{5.2}
\end{equation*}
$$

where $c$ is the propagation speed of the electromagnetic wave. Because the distance difference $\Delta r$ can be obtained by measuring TDOA $\Delta t$ multiplied by constant $c$, expression (5.2) is the TDOA measurement expression.
By differentiating expression (5.2), one can obtain

$$
\begin{equation*}
\Delta \dot{r}=c \Delta \dot{t}=\dot{r}_{1}-\dot{r}_{2}=\Delta v_{r} \tag{5.3}
\end{equation*}
$$



Figure 5.4 Sketch of 3D of dual-satellite geolocation
where the change rate of TDOA $\Delta \dot{i}$ can be indirectly measured through Doppler FDOA $\Delta f_{d}$, and their relationship is as follows:

$$
\begin{equation*}
\Delta f_{d}=-f_{0} \Delta \dot{t} \tag{5.4}
\end{equation*}
$$

where $f_{0}$ is the frequency of the signal from the emitter. Therefore, the FDOA measurement expression can be obtained:

$$
\begin{equation*}
\Delta \dot{r}=-\frac{c}{f_{0}} \Delta f_{d}=-\Delta f_{d} \lambda, \tag{5.5}
\end{equation*}
$$

where $\lambda=c / f_{0}$ is the wavelength of the emitter signal.
The emitter is generally located on the earth's surface, which is also a prior information for geolocation. As for geolocation of the fixed emitter on the earth, the altitude is generally supposed to be known, that is, the emitter is located on the earth's surface with known radius, even though the earth's surface is not a spherical surface. It would be more accurate to use the average earth surface model defined by the WGS-84 coordinates system, but the method and conclusion obtained in this way are much closer to the method discussed in this section. For simplicity, a spherical surface is applied. Suppose the earth is a regular sphere with a radius of $R$; then the earth's surface is applicable to the expression:

$$
\begin{equation*}
\mathbf{u}^{\mathrm{T}} \mathbf{u}=R^{2} \tag{5.6}
\end{equation*}
$$

The location of the emitter can be accurately solved by combining the above three expressions (5.2), (5.5), and (5.6), but the solving process would be complex for such expressions are nonlinear. Finding the solution of the above expressions is the process of dual-satellite TDOA-FDOA geolocation.

Combine Equations (5.2), (5.3), and (5.6) and rewrite a system of equations as follows:

$$
\left.\begin{array}{l}
\Delta v_{r}=c \Delta \dot{t}=\dot{r}_{1}-\dot{r}_{2}  \tag{5.7}\\
\Delta r=c \Delta t=r_{1}-r_{2} \\
\mathbf{u}^{\mathrm{T}} \mathbf{u}=R^{2}
\end{array}\right\}
$$

Substitute the geocentric coordinate of satellites and the emitter into the above system of expressions, which yields

$$
\left.\begin{array}{l}
\frac{v_{x 1}\left(x-x_{1}\right)+v_{y 1}\left(y-y_{1}\right)+v_{z 1}\left(z-z_{1}\right)}{\sqrt{\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}+\left(z-z_{1}\right)^{2}}}-\frac{v_{x 2}\left(x-x_{2}\right)+v_{y 2}\left(y-y_{2}\right)+v_{z 2}\left(z-z_{2}\right)}{\sqrt{\left(x-x_{2}\right)^{2}+\left(y-y_{2}\right)^{2}+\left(z-z_{2}\right)^{2}}}=\Delta v_{r}  \tag{5.8}\\
\sqrt{\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}+\left(z-z_{1}\right)^{2}}-\sqrt{\left(x-x_{2}\right)^{2}+\left(y-y_{2}\right)^{2}+\left(z-z_{2}\right)^{2}}=\Delta r \\
x^{2}+y^{2}+z^{2}=R^{2}
\end{array}\right\} .
$$

In applying the dual-satellite geolocation method, the velocities of two satellites are always different whether two satellites are in or not in the same orbit, which increases the difficulty of the solving process. In general, in order to maintain relatively stable geolocation accuracy for a long time, two satellites are required to maintain a certain configuration. Therefore, two satellites often run in the same orbit or in two adjacent orbits. Compared with the radius of the earth, the distance between two satellites is very short, that is, the included angle between the lines of two satellites to the earth's center is very small (for example, if the distance between two satellites is 100 km , the corresponding included angle will be $0.77^{\circ}$ ). Therefore, as the velocity directions of the two satellites are almost the same, it can be approximately deemed that the moving velocities of the two satellites are the same, that is, $v_{x 1} \approx v_{x 2}=v_{x}, v_{y 1} \approx v_{y 2}=v_{y}$, and $v_{z 1} \approx v_{z 2}=v_{z}$. The system of geolocation equations (Equation (5.8)) can be rewritten as

$$
\left.\begin{array}{l}
\frac{v_{x}\left(x-x_{1}\right)+v_{y}\left(y-y_{1}\right)+v_{z}\left(z-z_{1}\right)}{\sqrt{\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}+\left(z-z_{1}\right)^{2}}}-\frac{v_{x}\left(x-x_{2}\right)+v_{y}\left(y-y_{2}\right)+v_{z}\left(z-z_{2}\right)}{\sqrt{\left(x-x_{2}\right)^{2}+\left(y-y_{2}\right)^{2}+\left(z-z_{2}\right)^{2}}}=\Delta v_{r}  \tag{5.9}\\
\sqrt{\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}+\left(z-z_{1}\right)^{2}}-\sqrt{\left(x-x_{2}\right)^{2}+\left(y-y_{2}\right)^{2}+\left(z-z_{2}\right)^{2}}=\Delta r \\
x^{2}+y^{2}+z^{2}=R^{2}
\end{array}\right\} .
$$

The usual method for solving the above nonlinear expressions is Newton's iteration method, but the stability of this algorithm is poor even though the numerical results can be obtained because the effect of iteration computation depends on the selection of an initial value point. Therefore, for the specific application of dual-satellite geolocation, an analytical algorithm is explored here to directly obtain the analytic solution of the system of expressions.

### 5.2.2 Solution Method of Algebraic Analysis

In 1997, Ho and Chan proposed an algebraic solution method [5] for the TDOA-FDOA dual-satellite geolocation problem, which is described as follows. Rearranging the second
equation of expression (5.7) yields

$$
\begin{equation*}
\Delta r^{2}+2 \Delta r r_{1}+r_{1}^{2}=r_{2}^{2} \tag{5.10}
\end{equation*}
$$

The following expression is obtained according to the definition of $r_{1}, r_{2}$ and the third equation of Equation (5.7):

$$
\begin{equation*}
r_{i}^{2}=R^{2}+\mathbf{s}_{i}^{\mathrm{T}} \mathbf{s}_{i}-2 \mathbf{s}_{i}^{\mathrm{T}} \mathbf{u} \quad(i=1,2) \tag{5.11}
\end{equation*}
$$

Substituting this expression into the third equation of Equation (5.9) yields

$$
\begin{equation*}
\Delta r^{2}+2 \Delta r r_{1}=\mathbf{s}_{2}^{\mathrm{T}} \mathbf{s}_{2}-\mathbf{s}_{1}^{\mathrm{T}} \mathbf{s}_{1}-2\left(\mathbf{s}_{2}-\mathbf{s}_{1}\right)^{\mathrm{T}} \mathbf{u} \tag{5.12}
\end{equation*}
$$

Differentiate both sides of expression (5.12) with respect to the time to obtain an equivalent expression to the first equation of expression (5.7):

$$
\begin{equation*}
2 \Delta r \Delta \dot{r}+2 \Delta r \dot{r}_{1}+2 \Delta \dot{r} r_{1}-2 \mathbf{s}_{2}^{\mathrm{T}} \dot{\mathbf{s}}_{2}+2 \mathbf{s}_{1}^{\mathrm{T}} \dot{\mathbf{s}}_{1}=-2\left(\dot{\mathbf{s}}_{2}-\dot{\mathbf{s}}_{1}\right)^{\mathrm{T}} \mathbf{u} \tag{5.13}
\end{equation*}
$$

The following expression can be obtained according to expressions (5.11) to (5.13):

$$
\begin{equation*}
\mathbf{u}=\mathbf{G}_{1}^{-1} \mathbf{h}=\mathbf{G}_{4} \mathbf{r}_{1}+\mathbf{g}_{5} \dot{r}_{1}, \tag{5.14}
\end{equation*}
$$

where

$$
\begin{aligned}
\mathbf{G}_{1} & =-2\left[\begin{array}{c}
\mathbf{s}_{1}^{\mathrm{T}} \\
\mathbf{s}_{2}^{\mathrm{T}}-\mathbf{s}_{1}^{\mathrm{T}} \\
\dot{\mathbf{s}}_{2}^{\mathrm{T}}-\dot{\mathbf{s}}_{1}^{\mathrm{T}}
\end{array}\right], \\
\mathbf{h} & =\mathbf{G}_{2} \mathbf{r}_{1}+\mathbf{g}_{3} \dot{r}_{1}=\left[\begin{array}{ccc}
-R^{2}-\mathbf{s}_{1}^{\mathrm{T}} \mathbf{s}_{1} & 0 & 1 \\
\Delta r^{2}-\mathbf{s}_{2}^{\mathrm{T}} \mathbf{s}_{2}+\mathbf{s}_{1}^{\mathrm{T}} \mathbf{s}_{1} & 2 \Delta r & 0 \\
2 \Delta r \Delta \dot{r}-2 \mathbf{s}_{2}^{\mathrm{T}} \dot{\mathbf{s}}_{2}+2 \mathbf{s}_{1}^{\mathrm{T}} \dot{\mathbf{s}}_{1} & 2 \Delta \dot{r} & 0
\end{array}\right] \times\left[\begin{array}{c}
1 \\
r_{1} \\
r_{1}^{2}
\end{array}\right]+\left[\begin{array}{c}
0 \\
0 \\
2 \Delta r
\end{array}\right] \dot{r}_{1}, \\
\mathbf{G}_{4} & =\mathbf{G}_{1}^{-1} \mathbf{G}_{2}, \\
\mathbf{g}_{5} & =\mathbf{G}_{1}^{-1} \mathbf{g}_{3}, \\
\mathbf{r}_{1} & =\left[\begin{array}{lll}
1 & r_{1} & r_{1}^{2}
\end{array}\right]^{\mathrm{T}} .
\end{aligned}
$$

Expression (5.14) is equivalent to expression (5.7). Next, the solution of expression (5.14) is introduced.
By differentiating expression (5.1), we can obtain

$$
\begin{equation*}
\dot{r}_{i} r_{i}=\left(\mathbf{s}_{i}-\mathbf{u}\right)^{\mathrm{T}} \dot{\mathbf{s}}_{i} \quad(i-1,2) . \tag{5.15}
\end{equation*}
$$

Substitute expression (5.14) into expression (5.15) and set $i=1$ :

$$
\begin{equation*}
\dot{r}_{1}=\frac{1}{r_{1}+p} \mathbf{g}_{6}^{\mathrm{T}} \mathbf{r}_{1}, \tag{5.16}
\end{equation*}
$$

where

$$
p=\dot{\mathbf{s}}_{1}^{\mathrm{T}} \mathbf{g}_{5} \quad \text { and } \quad \mathbf{g}_{6}=\left[\begin{array}{c}
\mathbf{s}_{1}^{\mathrm{T}} \dot{\mathbf{s}}_{1} \\
0 \\
0
\end{array}\right]-\mathbf{G}_{4}^{\mathrm{T}} \dot{\mathbf{s}}_{1} .
$$

Substitute expression (5.16) into expression (5.14) to derive

$$
\begin{equation*}
\mathbf{u}=\frac{\mathbf{G}_{7} \mathbf{r}_{2}}{r_{1}+p}, \tag{5.17}
\end{equation*}
$$

where

$$
\mathbf{G}_{7}=\left[\begin{array}{ll}
p \mathbf{G}_{4}+\mathbf{g}_{5} \mathbf{g}_{6}^{\mathrm{T}} & \mathbf{0}_{3 \times 1}
\end{array}\right]+\left[\begin{array}{ll}
\mathbf{0}_{3 \times 1} & \mathbf{G}_{4}
\end{array}\right] \quad \text { and } \quad \mathbf{r}_{2}=\left[\begin{array}{llll}
1 & r_{1} & r_{1}^{2} & r_{1}^{3}
\end{array}\right]^{\mathrm{T}} .
$$

Substitute the above expression into the third expression, $\mathbf{u}^{\mathrm{T}} \mathbf{u}=R^{2}$, of Equation (5.17), which yields

$$
\begin{equation*}
\mathbf{g}_{8}^{\mathrm{T}} \cdot \mathbf{r}_{3}=\mathbf{0}, \tag{5.18}
\end{equation*}
$$

where

$$
\begin{aligned}
& \mathbf{g}_{8}=\left(\begin{array}{l}
\mathbf{G}_{8}(1,1)-\mathbf{g}_{7}(1) \\
\mathbf{G}_{8}(2,1)+\mathbf{G}_{8}(1,2)-\mathbf{g}_{7}(2) \\
\mathbf{G}_{8}(3,1)+\mathbf{G}_{8}(2,2)+\mathbf{G}_{8}(1,3)-\mathbf{g}_{7}(3) \\
\mathbf{G}_{8}(4,1)+\mathbf{G}_{8}(3,2)+\mathbf{G}_{8}(2,3)+\mathbf{G}_{8}(1,4) \\
\mathbf{G}_{8}(4,2)+\mathbf{G}_{8}(3,3)+\mathbf{G}_{8}(2,4) \\
\mathbf{G}_{8}(4,3)+\mathbf{G}_{8}(3,4) \\
\mathbf{G}_{8}(4,4)
\end{array}\right), \\
& \mathbf{G}_{8}=\mathbf{G}_{7}^{\mathrm{T}} \mathbf{G}_{7}, \\
& \mathbf{g}_{7}=R^{2}\left[\begin{array}{llll}
p^{2} & 2 p & 1 & 0
\end{array}\right]^{\mathrm{T}}, \\
& \mathbf{r}_{3}=\left[\begin{array}{lllllll}
1 & r_{1} & r_{1}^{2} & r_{1}^{3} & r_{1}^{4} & r_{1}^{5} & r_{1}^{6}
\end{array}\right]^{\mathrm{T}} .
\end{aligned}
$$

Expression (5.18) is a polynomial equation with respect to $r_{1}^{6}$. Such expressions have no elementary solution from the point of view of algebraic theory, while a numerical solution can be constructed.
Express the polynomial equation (Equation (5.18)) with respect to $r_{1}$ in the following form:

$$
\begin{equation*}
a_{6} r_{1}^{6}+a_{5} r_{1}^{5}+a_{4} r_{1}^{4}+a_{3} r_{1}^{3}+a_{2} r_{1}^{2}+a_{1} r_{1}+a_{0}=0 \tag{5.19}
\end{equation*}
$$

Define the companion matrix $\mathbf{A}$ as follows:

$$
\mathbf{A}=\left(\begin{array}{cccccc}
-a_{5} / a_{6} & -a_{4} / a_{6} & -a_{3} / a_{6} & -a_{2} / a_{6} & -a_{1} / a_{6} & -a_{0} / a_{6}  \tag{5.20}\\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right) .
$$

The eigenvalue of matrix $\mathbf{A}$ is the same as the solution of expression (5.19). The eigenvalue of the matrix can easily be calculated using the numerical method. Certainly, the classical Newton iteration method can also be used to solve expression (5.19) to obtain $r_{1}$, and by substituting $r_{1}$ into expression (5.14) to obtain the geolocation solution.
No matter what kind of method is adopted to solve expression (5.19), six solutions will be obtained but only one will be the true solution with respect to the geolocation. Therefore, the ambiguity of solutions is caused. These ambiguous geolocation solutions can be divided into
two categories. One comprises ambiguous geolocation solutions as a result of the extraneous roots introduced in the expression solving process, which can be removed by verifying the root of these expressions. The other comprises ambiguous geolocation solutions introduced due to the existence of multiple points of intersection on the geolocation surface of dual-satellite geolocation, which can be removed through system design, as described in Section 5.2.4.
In the derivation of the algorithm, pay attention to the fact that expression (5.14) requires matrix $\mathbf{G}_{1}$ to be invertible. By observing $\mathbf{G}_{1}$, it can be invertible when:

1. Satellite 1 , satellite 2 and the earth center are not in a same line, which is satisfied in most cases.
2. The direction of the relative velocity of satellite 2 and satellite 1 is neither in the line between the earth center and satellite 1 satellite 2 , nor the line between satellite 1 and satellite 2 .

Condition 2 means that if two satellites are in the same orbit and are very close, the $\mathbf{G}_{1}$ matrix would be close to being not invertible and its geolocation effect would be poor.

### 5.2.3 Approximate Analytical Method for Same-Orbit Satellites

When two satellites running in the same orbit or in two adjacent orbits are close to each other, their velocities are very close to each other or can be approximately deemed as being the same. In this case, the $\mathbf{G}_{1}$ matrix is close to being not invertible and its geolocation accuracy would be poor. The problem of a dual-satellite in this case is called the same-orbit satellites geolocation problem. (When two satellites are in the same orbit, if their distance is long, the velocity difference will be large. However, it is not applicable to cooperative geolocation due to the small common coverage area of satellites. Therefore, such a circumstance is not under consideration.)
Based on the analysis of the previous section, when the velocities of two satellites are the same, geolocation expression (5.8) can be simplified to expression (5.9). Methods of solving the system of nonlinear expression (5.9) are discussed below.
Rewrite expression (5.1) as follows:

$$
\begin{align*}
& r_{1}=\sqrt{\left(x-x_{1}\right)^{2}+\left(y-y_{1}\right)^{2}+\left(z-z_{1}\right)^{2}}  \tag{5.21}\\
& r_{2}=\sqrt{\left(x-x_{2}\right)^{2}+\left(y-y_{2}\right)^{2}+\left(z-z_{2}\right)^{2}} \tag{5.22}
\end{align*}
$$

Substituting the above two expressions into the second expression of (5.9) yields

$$
\begin{equation*}
\left(x_{1}-x_{2}\right) x+\left(y_{1}-y_{2}\right) y+\left(z_{1}-z_{2}\right) z=k_{1}-r_{1} \Delta r \tag{5.23}
\end{equation*}
$$

where

$$
\begin{equation*}
k_{1}=\frac{1}{2}\left[\left(x_{1}^{2}+y_{1}^{2}+z_{1}^{2}\right)-\left(x_{2}^{2}+y_{2}^{2}+z_{2}^{2}\right)+\Delta r^{2}\right] \tag{5.24}
\end{equation*}
$$

Square Equation (5.21) and then subtract it from the third expression of Equation (5.9):

$$
\begin{equation*}
x_{1} x+y_{1} y+z_{1} z=k_{2}-\frac{1}{2} r_{1}^{2}, \tag{5.25}
\end{equation*}
$$

where

$$
\begin{equation*}
k_{2}=\frac{1}{2}\left[x_{1}^{2}+y_{1}^{2}+z_{1}^{2}+R^{2}\right] . \tag{5.26}
\end{equation*}
$$

Rewrite the first expression in the system of expression (5.9) as

$$
\begin{equation*}
\frac{v_{x}\left(x-x_{1}\right)+v_{y}\left(y-y_{1}\right)+v_{z}\left(z-z_{1}\right)}{r_{1}}-\frac{v_{x}\left(x-x_{2}\right)+v_{y}\left(y-y_{2}\right)+v_{z}\left(z-z_{2}\right)}{r_{1}-\Delta r}=\Delta v_{r} \tag{5.27}
\end{equation*}
$$

Rearranging it then yields

$$
\begin{equation*}
v_{x} x+v_{y} y+v_{z} z=k_{3}+k_{4} r_{1}+k_{5} r_{1}^{2} \tag{5.28}
\end{equation*}
$$

where

$$
\begin{aligned}
& k_{3}=v_{x} x_{1}+v_{y} y_{1}+v_{z} z_{1}, \quad k_{4}=\Delta v_{r}-\left[v_{x}\left(x_{1}-x_{2}\right)+v_{y}\left(y_{1}-y_{2}\right)+v_{z}\left(z_{1}-z_{2}\right)\right] / \Delta r, \quad \text { and } \\
& k_{5}=-\Delta v_{r} / \Delta r .
\end{aligned}
$$

Solve the system of linear expressions of Equations (5.23), (5.25), and (5.27), which yields

$$
\begin{equation*}
\mathbf{A} \mathbf{u}=\mathbf{f} \tag{5.29}
\end{equation*}
$$

where

$$
\mathbf{A}=\left[\begin{array}{ccc}
x_{1}-x_{2} & y_{1}-y_{2} & z_{1}-z_{2} \\
x_{1} & y_{1} & z_{1} \\
v_{x} & v_{y} & v_{2}
\end{array}\right] \quad \text { and } \quad \mathbf{f}=\left[\begin{array}{c}
k_{1}-r_{1} \Delta r \\
k_{2}-\frac{1}{2} r_{1}^{2} \\
k_{3}+k_{4} r_{1}+k_{5} r_{1}^{2}
\end{array}\right]=\left[\begin{array}{l}
f_{1} \\
f_{2} \\
f_{3}
\end{array}\right] .
$$

The following discusses the two cases of $\mathbf{A}$ invertible and $\mathbf{A}$ not invertible:

1. Matrix A invertible

Directly solve the expression (5.29) to yield

$$
\begin{equation*}
\widehat{\mathbf{u}}=\mathbf{A}^{-1} \mathbf{f} \tag{5.30}
\end{equation*}
$$

Substitute Equation (5.29) into the third expression of Equation (5.9) to obtain a quartic expression with $r_{1}$ as the unknown number:

$$
\begin{equation*}
s_{1} r_{1}^{4}+s_{2} r_{1}^{3}+s_{3} r_{1}^{2}+s_{4} r_{1}+s_{5}=0 \tag{5.31}
\end{equation*}
$$

Solve the quartic expression (5.31) to find the root of $r_{1}$, and then substitute this into expression (5.30) to get $(x, y, z)$. In fact, eight groups of solutions can be obtained by using the above method to solve the expression, and only four groups are consistent with the system of geolocation expression (5.9), while the other four groups are the extraneous roots generated in the expression solving process. In four groups of geolocation solutions, only one group is the location of the emitter and the others are the imaginary roots. Therefore, operations of root verification and solving geolocation ambiguity are required.
2. Matrix A not invertible

By observing matrix $\mathbf{A}$, the corresponding physical situations of $\mathbf{A}$ that are not invertible are as follows:
a. The direction of the velocity of two satellites is the same as that of the line between the earth center and any satellite.
b. The direction of the velocity of two satellites is the same as that of the line between them.

As for the first situation, two satellites have radial motion relative to the earth center, which does not exist in actual application. As for the latter, the location of the emitter can still be obtained for satellites at a certain height. Express the second case of A not invertible as follows:

$$
\begin{equation*}
\frac{v_{x}}{x_{1}-x_{2}}=\frac{v_{y}}{y_{1}-y_{2}}=\frac{v_{z}}{z_{1}-z_{2}}=K \tag{5.32}
\end{equation*}
$$

Substitute the above expression into expressions (5.23) and (5.28), which yields

$$
\begin{equation*}
K\left(k_{1}-r_{1} \Delta r\right)=k_{3}+k_{4} r_{1}+k_{5} r_{1}^{2} . \tag{5.33}
\end{equation*}
$$

Solving the above quadratic expression with respect to $r_{1}$ yields two roots:

$$
\begin{equation*}
\widehat{r}_{1}=\frac{-\left(k_{4}+K \Delta r\right) \pm \sqrt{\left(k_{4}+K \Delta r\right)^{2}-4 k_{5}\left(k_{3}-K k_{1}\right)}}{2 k_{5}} \tag{5.34}
\end{equation*}
$$

Substitute Equation (5.34) into Equation (5.32), with expression (5.25) and the third expression in the system of expression (5.9), and solve these three equations to obtain the solution of the emitter location $(x, y, z)$. Suppose $x$ is known; the following expression can then be obtained according to expressions (5.23) and (5.25):

$$
\mathbf{B}\left[\begin{array}{l}
y  \tag{5.35}\\
z
\end{array}\right]=\mathbf{G},
$$

where

$$
\mathbf{B}=\left[\begin{array}{cc}
y_{1}-y_{2} & z_{1}-z_{2} \\
y_{1} & z_{1}
\end{array}\right] \quad \text { and } \quad \mathbf{G}=\left[\begin{array}{l}
f_{1} \\
f_{2}
\end{array}\right]-\left[\begin{array}{c}
\left(x_{1}-x_{2}\right) x \\
x_{1} x
\end{array}\right] .
$$

If two satellites and the earth center are not in the same line and $\mathbf{B}$ is invertible, the following expression can be obtained:

$$
\left[\begin{array}{l}
y  \tag{5.36}\\
z
\end{array}\right]=\mathbf{B}^{-1} \mathbf{G}
$$

Substitute $y$ and $z$ into the third expression of (5.9) and solve a quadratic expression with respect to $x$ :

$$
\begin{equation*}
p x^{2}+n x+m=0 \tag{5.37}
\end{equation*}
$$

Therefore, the following expression can be obtained:

$$
\begin{equation*}
x=\frac{-n \pm \sqrt{n^{2}-4 p m}}{2 p} \tag{5.38}
\end{equation*}
$$

For each $r_{1}$, two solutions can be obtained. Substitute them into expression (5.36) to obtain $y$ and $z$. The true solution $(x, y, z)$ can be obtained after removal of the imaginary roots.

### 5.2.4 Method for Eliminating an Ambiguous Geolocation Point

According to the observation of the above equations solving process, the two real solutions $\mathbf{u}$ and $\mathbf{u}^{\prime}$ obtained in dual-satellite geolocation are symmetrical on both sides of the line connecting the satellites with the subsatellite point, which cannot be distinguished when


Figure 5.5 Method for solving ambiguity in dual-satellite geolocation
only based on TDOA and FDOA from mathematics. Other information should be introduced to make the distinction. For example, make the main lobe of the receiving antenna of the satellite at the left side or the right side and the ambiguous geolocation point $\mathbf{u}^{\prime}$ will be in the direction of the other side. In such a situation, the signals transmitted from the ambiguous point cannot reach the receiver of the satellite. Alternatively, set the right and left antennas for the primary satellite and then compare the amplitudes of the right and left antennas to roughly determine the direction of arrival, so as to remove the ambiguous point in the geolocation solving process. The principle of this scheme is shown in Figure 5.5.
However, this method is not applicable for the emitter near the subsatellite point, due to the large error of TDOA-FDOA geolocation, which causes great uncertainty for direction determination.

### 5.3 Error Analysis for TDOA-FDOA Geolocation

The actual dual-satellite system measurement is always affected by noise, which definitely causes error to the measurement of TDOA, FDOA, and location. Therefore, influences of such error on the TDOA-FDOA geolocation system must be analyzed.

### 5.3.1 Analytic Method for the Geolocation Error

Differentiate the second expression of Equation (5.9) at the emitter point $(x, y, z)$, which yields

$$
\begin{align*}
& \left(g_{x 1}-g_{x 2}\right) \mathrm{d} x+\left(g_{y 1}-g_{y 2}\right) \mathrm{d} y+\left(g_{z 1}-g_{z 2}\right) \mathrm{d} z \\
& \quad=\mathrm{d} \Delta r+\left(g_{x 1} \mathrm{~d} x_{1}-g_{x 2} \mathrm{~d} x_{2}\right)+\left(g_{y 1} \mathrm{~d} y_{1}-g_{y 2} \mathrm{~d} y_{2}\right)+\left(g_{z 1} \mathrm{~d} z_{1}-g_{z 2} \mathrm{~d} z_{2}\right) \tag{5.39}
\end{align*}
$$

Where

$$
g_{s i}=\frac{s-s_{i}}{r_{i}}(s=x, y, z, \quad i=1,2)
$$

Differentiate the first expression of Equation (5.9) at the emitter point ( $x, y, z$ ), which yields

$$
\begin{align*}
&\left(c_{x 1}-c_{x 2}\right) \mathrm{d} x+\left(c_{y 1}-c_{y 2}\right) \mathrm{d} y+\left(c_{z 1}-c_{z 2}\right) \mathrm{d} z \\
&= \mathrm{d} \Delta v_{r}+\left(c_{x 1} \mathrm{~d} x_{1}-c_{x 2} \mathrm{~d} x_{2}\right)+\left(c_{y 1} \mathrm{~d} y_{1}-c_{y 2} \mathrm{~d} y_{2}\right)+\left(c_{z 1} \mathrm{~d} z_{1}-c_{z 2} \mathrm{~d} z_{2}\right) \\
& \quad+\left(g_{x 1}-g_{x 2}\right) \mathrm{d} v_{x}+\left(g_{y 1}-g_{y 2}\right) \mathrm{d} v_{y}+\left(g_{z 1}-g_{z 2}\right) \mathrm{d} v_{z} \tag{5.40}
\end{align*}
$$

where

$$
\begin{aligned}
& c_{s i}=\frac{v_{s} r_{i}-\Delta v_{r_{i}}\left(s-s_{i}\right)}{r_{i}^{2}} \quad(s=x, y, z, i=1,2), \\
& v_{r i}=\left[v_{x}\left(x-x_{i}\right)+v_{y}\left(y-y_{i}\right)+v_{z}\left(z-z_{i}\right)\right] / r_{i} .
\end{aligned}
$$

Differentiate the third expression of Equation (5.9) at the emitter point $(x, y, z)$, which yields

$$
\begin{equation*}
x \mathrm{~d} x+y \mathrm{~d} y+z \mathrm{~d} z=R \mathrm{~d} H \tag{5.41}
\end{equation*}
$$

where $\mathrm{d} H=\mathrm{d} R$, the error of the earth radius, is equivalent to the error of the emitter altitude.
Rearrange the above expressions (5.39) to (5.41) into matrix form:

$$
\begin{equation*}
\mathbf{C d} \mathbf{u}=\mathrm{d} \mathbf{z}+\mathbf{U d} \mathbf{s}_{1}-\mathbf{W} \mathrm{ds}_{2}+\mathbf{V d} \mathbf{v} \tag{5.42}
\end{equation*}
$$

where

$$
\begin{aligned}
& \mathbf{C}=\left(\begin{array}{ccc}
c_{x 1}-c_{x 2} & c_{y 1}-c_{y 2} & c_{z 1}-c_{z 2} \\
g_{x 1}-g_{x 2} & g_{y 1}-g_{y 2} & g_{z 1}-g_{z 2} \\
x & y & z
\end{array}\right), \mathbf{U}=\left(\begin{array}{ccc}
c_{x 1} & c_{y 1} & c_{z 1} \\
g_{x 1} & g_{y 1} & g_{z 1} \\
0 & 0 & 0
\end{array}\right), \\
& \mathbf{W}=\left(\begin{array}{ccc}
c_{x 2} & c_{y 2} & c_{z 2} \\
g_{x 2} & g_{y 2} & g_{z 2} \\
0 & 0 & 0
\end{array}\right), \mathbf{V}=\left(\begin{array}{ccc}
g_{x 1}-g_{x 2} & g_{y 1}-g_{y 2} & g_{z 1}-g_{z 2} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \\
& \mathrm{d} \mathbf{z}=\left(\begin{array}{c}
\mathrm{d} \Delta v_{r} \\
\mathrm{~d} \Delta r \\
R \mathrm{~d} H
\end{array}\right), \mathrm{d} \mathbf{s}_{1}=\left(\begin{array}{l}
\mathrm{d} x_{1} \\
\mathrm{~d} y_{1} \\
\mathrm{~d} z_{1}
\end{array}\right), \mathrm{d} \mathbf{s}_{2}=\left(\begin{array}{l}
\mathrm{d} x_{2} \\
\mathrm{~d} y_{2} \\
\mathrm{~d} z_{2}
\end{array}\right), \mathrm{d} \mathbf{v}=\left(\begin{array}{l}
\mathrm{d} v_{x} \\
\mathrm{~d} v_{y} \\
\mathrm{~d} v_{z}
\end{array}\right) .
\end{aligned}
$$

When measuring the position of two satellites, measure the relative location between satellites (baseline) to obtain a more accurate relative location of the satellites, that is, $\mathbf{s}_{2}=\mathbf{s}_{1}+\mathbf{l}$, where $\left[\begin{array}{ll}x_{2}-x_{1} & y_{2}-y_{1} \\ z_{2} & -z_{1}\end{array}\right]^{\mathrm{T}}$ is the vector of the satellite baseline. In this way, $\mathrm{d} \mathbf{s}_{2}=$ $\mathrm{ds} \mathbf{1}_{1}+\mathrm{dl}$ can be obtained. Therefore, the above expression can be written as

$$
\begin{equation*}
\mathbf{C d} \mathbf{u}=\mathrm{d} \mathbf{z}+(\mathbf{U}-\mathbf{W}) \mathrm{d} \mathbf{s}_{2}+\mathbf{W} \mathrm{d} \mathbf{l}+\mathbf{V} \mathrm{d} \mathbf{v} \tag{5.43}
\end{equation*}
$$

Suppose that all errors are zero-mean and Gaussian distributed, are uncorrelated, the measurement covariance matrix is $\mathbf{R}_{z}=E\left[\mathrm{~d} \mathbf{z} \mathbf{z}^{\mathrm{T}}\right]$, the covariance matrix of the satellite location error is $\mathbf{R}_{s}=E\left[\mathrm{ds}_{2} \mathrm{~d} \mathbf{s}_{2}^{\mathrm{T}}\right]$, the covariance matrix of the satellite velocity error is $\mathbf{R}_{v}=E\left[\mathrm{~d} \mathbf{v} \mathrm{~d} \mathbf{v}^{\mathrm{T}}\right]$,


Figure 5.6 The mean square error (MSE) and theoretical error when the distance between satellites $=50 \mathrm{~km}, \sigma_{\Delta t}=80 \mathrm{~ns}$, and $\sigma_{\Delta f_{d}}=10 \mathrm{~Hz}$. (a) GDOP of the MSE obtained through Monte Carlo simulation statistics and (b) theoretic GDOP of the geolocation error
and the covariance matrix of the satellite baseline error is $\mathbf{R}_{l}=E\left[\mathrm{dl} \mathrm{dl}{ }^{\mathrm{T}}\right]$. Then the geolocation covariance matrix in the ECEF coordinates system will be

$$
\begin{equation*}
\mathbf{P}_{\mathrm{du}}=E\left[\mathrm{~d} \mathbf{u} \mathrm{~d} \mathbf{u}^{\mathrm{T}}\right]=\mathbf{C}^{-1}\left(\mathbf{R}_{z}+(\mathbf{U}-\mathbf{W}) \mathbf{R}_{s}(\mathbf{U}-\mathbf{W})^{\mathrm{T}}+\mathbf{W} \mathbf{R}_{l} \mathbf{W}^{\mathrm{T}}+\mathbf{V} \mathbf{R}_{v} \mathbf{V}^{\mathrm{T}}\right)\left(\mathbf{C}^{-1}\right)^{\mathrm{T}} . \tag{5.44}
\end{equation*}
$$

The range error of geolocation is

$$
\begin{equation*}
\operatorname{GDOP}(x, y, z)=\sqrt{\operatorname{tr}\left(\mathbf{P}_{\mathrm{du}}\right)} \tag{5.45}
\end{equation*}
$$

where $\operatorname{tr}(\cdot)$ represents the matrix trace.

### 5.3.2 GDOP of the Dual LEO Satellite Geolocation Error

In the dual-satellite geolocation system, the velocity direction of the two satellites and the direction of the line between the two satellites will form different angles at different times (if satellites are in different orbits), and, accordingly, the geolocation accuracy distribution of the system will be different. Next, three cases will be discussed where the angle between the velocity direction of the satellites and their connecting line is $90^{\circ}, 0^{\circ}$, and between $0^{\circ}$ and $90^{\circ}$, to observe the geolocation error distribution:
I. Geolocation accuracy when the velocity direction of satellites is vertical to their connecting line (suppose two satellites are in different orbits).

Suppose that the subsatellite points of satellite 1 and satellite 2 are as shown by ' $\oplus 1$ ' and ' $\oplus 2$ ' in the center of the parts of Figure 5.6. Assume that the distance between the two satellites is 50 km , the orbital altitude of satellite is 700 km , the direction of the satellite motion is vertical to their connecting line, and the measurements of TDOA and FDOA are zero-mean Gaussian white noise with $\sigma_{\Delta t}, \sigma_{\Delta f_{d}}$ as their root mean square error
(RMSE). Adopting the method described in Section 5.2.3 to calculate the location, using the method of the Monte Carlo test, and supposing the test is repeated 100 times, the theoretic geolocation error near the subsatellite point on the earth's surface and GDOP (geometric dilution of precision) of the RMSE can be obtained, as shown in Figure 5.6.

By comparing Figure 5.6a and b, it can be observed that the GDOP of geolocation accuracy obtained through Monte Carlo simulation statistics is almost the same as the theoretic GDOP, which verifies the correctness of theoretic GDOP.

In this simulation, from Figure 5.6 it can be observed that:

1. Two sides of the subsatellite track have a better geolocation effect, and the geolocation error can be lower than 2 km even in a large area.
2. In the direction of satellite motion, the geolocation error is relatively greater, which is consistent with the analysis in Section 5.1. The reason is that the CEP (circular error probability) error is geometrically magnified and is tangent to the earth's surface at this moment.
3. There is an area with a large geolocation error both 700 km above or below the satellite at the same latitude, because the emitter cannot be located when it is in the direction of the line between the satellite and the subsatellite points. However, in 3D geolocation, such an unobservable area is the area on both sides of the line connecting the two satellites and its location is related to the altitude of satellites.
In fact, in the area both 700 km above and below the satellite at the same latitude in Figure 5.6, the geolocation error is large, which is caused by satellite altitude and earth curvature. GDOPs of the geolocation error will be obtained if the altitude of the satellite is changed, as shown in Figure 5.7.

By comparing Figure 5.7 a to d , it can be seen that when the satellite is at a low altitude, the matter of 3D geolocation can be construed as the matter of 2D geolocation. The GDOP near the subsatellite point of the satellite conforms to the above law of 2D unobservability. However, with the increase of satellite altitude, the unobservable area in the direction of the line between the satellite and the subsatellite points will be gradually divided into two parts. Therefore, the two unobservable areas with the same latitude in Figure 5.6 are caused by the satellite altitude and the earth's surface.
II. Geolocation accuracy and observability when the velocity direction of satellites and their connecting line are the same (in the same orbit). If the direction of satellite motion is changed to be in the same direction as the connecting line, the theoretic GDOP in Section 5.3.1 can be obtained, as shown in Figure 5.8.

From Figure 5.8 the following can be noted:

1. Two sides of the satellite orbit have better geolocation accuracy, and the geolocation error can be lower than 5 km , even in a large area.
2. The geolocation error in the direction of satellite motion is great, which is the same as shown in Figure 5.6.
3. There is no area with as great a geolocation error, both 1000 km above or below the satellite, as the result shown in Figure 5.6.
In fact, if the two satellites run in the same orbit, the direction of satellite motion can always be maintained to be the same as the line between the satellites, even though the time when they arrive at perigee is a little different. In this way, the unobservable area in the same latitude in Figure 5.6 can be eliminated.


Figure 5.7 GDOP when the distance between satellites $=50 \mathrm{~km}, \sigma_{\Delta t}=80 \mathrm{~ns}$, and $\sigma_{\Delta f_{d}}=10 \mathrm{~Hz}$. (a) $H=50 \mathrm{~km}$, (b) $H=100 \mathrm{~km}$, (c) $H=300 \mathrm{~km}$, and (d) $H=600 \mathrm{~km}$


Figure 5.8 GDOP when the distance between satellites $=50 \mathrm{~km}, \sigma_{\Delta t}=80 \mathrm{~ns}$, and $\sigma_{\Delta f_{d}}=10 \mathrm{~Hz}$
III. Geolocation accuracy and observability when the velocity direction of the satellite is neither the same as nor vertical to their connecting line. If the direction of satellite motion is neither vertical to nor the same as the connection line, and supposing that there is a certain angle $\theta$ instead between them, theoretic GDOPs under different $\theta$ can be obtained, as shown in Figure 5.9.

By comparing Figures 5.6b, 5.8, and 5.9a to c, the influence of the satellite motion direction change on the GDOP distribution can be observed. If the direction of satellite motion is different from the line between the satellites, the unobservable geolocation area is neither in the direction of satellite motion nor in the direction of the line between the satellites.

Summarizing the analysis of this section, the following conclusions can be obtained:

1. When the velocity direction of the satellite is vertical to the line between the satellites, the observability in the area of velocity direction of the satellite is worse. In addition, with the change of satellite altitude, there exists an unobservable strip area at a certain distance from and parallel to the lines between the subsatellite points. Such strip areas represent longitudinal symmetry. However, for the area in the middle of such strip areas, geolocation accuracy is good.


Figure 5.9 GDOP when the distance between satellites $=50 \mathrm{~km}, \sigma_{\Delta t}=80 \mathrm{~ns}$, and $\sigma_{\Delta f_{d}}=10 \mathrm{~Hz}$. (a) $\theta=22.5^{\circ}$, (b) $\theta=45^{\circ}$, and (c) $\theta=67.5^{\circ}$
2. When the velocity direction of the satellite is the same as the line between the satellites, the observability in the velocity direction of the satellite (also the direction of the line between the satellites) is poor. The geolocation accuracy in the large area on both sides of the line between the satellites is good.
3. When the velocity direction of the satellite is neither vertical to nor the same as the line between the satellites, the area with a large geolocation error (i.e., poor observability) is expanded in the form of a strip in the middle of the satellite, and is not in the direction of satellite motion nor the line between the satellites.

From the above conclusions, we prefer to select the case when two satellites run in the same orbit, so the same orbit distribution should be adopted as much as possible for dual LEO satellites TDOA-FDOA geolocation. Next, we will analyze the relationship between the geolocation distribution of satellites running in the same orbit and the parameter accuracy.


Figure 5.10 GDOP when the distance between satellites $=50 \mathrm{~km}, \sigma_{\Delta t}=80 \mathrm{~ns}$, and $\sigma_{\Delta f_{d}}=10 \mathrm{~Hz}$. (a) $H=1000 \mathrm{~km}$ and (b) $H=300 \mathrm{~km}$


Figure 5.11 GDOP when $\sigma_{\Delta t}=80 \mathrm{~ns}, \sigma_{\Delta f_{d}}=10 \mathrm{~Hz}, \delta l=50 \mathrm{~m}, \delta x=150 \mathrm{~m}, \delta v=1 \mathrm{~m} / \mathrm{s}$, and $\delta H=10 \mathrm{~m}$. (a) Distance between satellites $=200 \mathrm{~km}$ and (b) distance between satellites $=50 \mathrm{~km}$

### 5.3.3 Analysis of Various Factors Influencing GDOP

When two satellites run in the same orbit, the velocity direction of the satellite is consistent with the direction of the line between the satellites. The following is an analysis of the influence of error factors on the geolocation error under typical scenarios. Such factors are satellite altitude, distance between satellites, TDOA measurement error, FDOA measurement error, velocity measurement error, altitude assumption error, measurement error of distance between satellites, and so on.

1. Influence of satellite altitude on geolocation accuracy

When two satellites run in the same orbit, the direction of satellite motion can be maintained the same as the direction of the lines between them, even though the time when they arrive at the perigee is a little different.

Set the distance between satellites as 50 km , the TDOA measurement error RMS (root mean square) $\sigma_{\Delta t}=80 \mathrm{~ns}$, and the FDOA measurement error RMS $\sigma_{\Delta f_{d}}=10 \mathrm{~Hz}$. What is shown in Figure 5.8 is the simulation accuracy and theoretical accuracy localized at a satellite altitude of 700 km .

The GDOPs localized when the satellite altitude is changed to 1000 and 300 km are shown in Figure 5.10. In Figure 5.10a and b, for the geometric distribution of geolocation accuracy, the results are based on a theoretical calculation because the empirical error by statistical calculation is the same. By comparing Figure 5.8 with Figure 5.10a and b, the geolocation accuracy improves when the orbital altitude is decreasing.
2. Influence of distance between satellites on geolocation accuracy

For two satellites run in the same orbit at an altitude of 700 km , if the influence of such factors as the measurement error of satellite velocity $\delta v$, the ground altitude error $\delta H$, the measurement error of the satellite relative to position $\delta l$, and the measurement error of the primary satellite location $\delta x$ are taken into consideration, theoretic GDOPs as shown in Figure 5.11 will be calculated when the distances between the satellites are respectively 200 and 50 km and other conditions are the same as simulation 1.

By comparing and analyzing Figure 5.11a and b, increasing the distance between the satellites will effectively improve the geolocation accuracy.
3. Influence of TDOA measurement error on geolocation accuracy

When the satellite altitude is 700 km , the distance between the satellites is 50 km , the TDOA measurement error is changed to 40 and 200 ns , respectively, and other conditions are the same as simulation 1, the GDOPs shown in Figure 5.12 will be obtained.

By comparing and analyzing Figure 5.12, the GDOPs under different TDOA measurement accuracy are almost the same, which indicates that if the TDOA measurement error is less than 200 ns , its influence on the satellite geolocation error will be very small if $\sigma_{\Delta f_{d}}=10 \mathrm{~Hz}$ in this case.
4. Influence of FDOA measurement error on geolocation accuracy

When the satellite altitude is 1000 km , the distance between the satellites is 100 km , the FDOA measurement accuracy is changed into 1,5 , and 100 Hz , respectively, and other conditions are the same as simulation 1, the GDOPs shown in Figure 5.13 will be obtained.

By comparing and analyzing Figure 5.13a to c , the GDOPs under different FDOA measurement accuracy are quite different, which indicates that when $\sigma_{\Delta t}=80 \mathrm{~ns}$, the influence of the FDOA measurement error on the satellite geolocation error is great.

Therefore, the key point in dual-satellite TDOA-FDOA geolocation is to improve the FDOA measurement accuracy.
5. Influence of satellite velocity measurement error on geolocation accuracy

When the satellite altitude is 700 km , the distance between the satellites is 50 km , and other conditions are the same as simulation 1, the GDOPs are as shown in Figure 5.14, when the satellite velocity measurement error is changed to 0.1 and $10 \mathrm{~m} / \mathrm{s}$, respectively.

From Figure 5.14 we know that the satellite velocity measurement error has some influence on the geolocation error. When the velocity error is up to $\delta v=10 \mathrm{~m} / \mathrm{s}$, the geolocation accuracy decreases to some extent.
6. Influence of altitude error on geolocation accuracy

When the satellite altitude is 700 km , the distance between the satellites is 50 km , and other conditions are the same as simulation 1, the GDOPs when the emitter altitude assumption errors are 100 and 1000 m, respectively, shown in Figure 5.15 will be observed.

From Figure 5.15 we know that if the altitude assumption error is less than 1000 m , its influence on the geolocation error is small.

From the above analysis, we know that the error of dual LEO satellite geolocation is inversely proportional to the satellite altitude and the distance between satellites, and is directly proportional to the TDOA and FDOA error. The FDOA error affects satellite geolocation error most. Therefore, the key point of dual-satellite TDOA-FDOA geolocation is to improve the measurement accuracy of the FDOA.

### 5.4 Dual HEO Satellite TDOA-FDOA Geolocation

### 5.4.1 Dual Geosynchronous Orbit Satellites TDOA - FDOA Geolocation

Two satellites running in the geosynchronous orbit can also be used to realize TDOA-FDOA geolocation. Its typical application is to localize the jammer for the satellite broadcasting system. Two satellites used for dual HEO satellite geolocation are generally two geosynchronous


Figure 5.12 GDOP when the distance between satellites $=50 \mathrm{~km}, \sigma_{\Delta f_{d}}=10 \mathrm{~Hz}, \delta l=50 \mathrm{~m}, \delta x=150 \mathrm{~m}$, $\delta v=1 \mathrm{~m} / \mathrm{s}$, and $\delta H=10 \mathrm{~m}$. (a) $\sigma_{\Delta t}=40 \mathrm{~ns}$ and (b) $\sigma_{\Delta t}=200 \mathrm{~ns}$


Figure 5.13 GDOP when the distance between satellites $=50 \mathrm{~km}, \sigma_{\Delta t}=80 \mathrm{~ns}, \delta l=50 \mathrm{~m}, \delta x=150 \mathrm{~m}$, $\delta v=1 \mathrm{~m} / \mathrm{s}$, and $\delta H=10 \mathrm{~m}$. (a) $\sigma_{\Delta f_{d}}=1 \mathrm{~Hz}$, (b) $\sigma_{\Delta f_{d}}=5 \mathrm{~Hz}$, and (c) $\sigma_{\Delta f_{d}}=100 \mathrm{~Hz}$
or geostationary satellites close to each other. One is the primary satellite and the other one is the auxiliary satellite, which locates at the adjacent position and assists in geolocation. Normally, the main lobe of the beam of emitters (such as the user of a satellite communication system) aims at the primary satellite, and some side lobes of the emitter will point to the adjacent satellite (the auxiliary satellite) due to the wide beam-width antenna feature of the emitter. The two satellites will down-convert the uplink signals received and retransmit them to the ground station. The ground station within the coverage of the satellites may receive such signals, as shown in Figure 5.16.
Because the main lobe and side lobe of the same source signal differ only in power level and the propagation path of signals after being retransmitted by the two satellites is different, TDOA between the signals is generated. TDOA can be obtained by correlation of two received signals from the same source. For the location of two confirmed satellites, a hyperboloid revolution can be determined by TDOA with the locations of two satellites as its focus.
A synchronous satellite orbit is affected by the perturbation caused by the asymmetry of the earth's gravitational field, the gravitational field of the sun/moon, sunlight pressure, and so on,


Figure 5.14 GDOP when the distance between satellites $=50 \mathrm{~km}, \sigma_{\Delta t}=80 \mathrm{~ns}, \sigma_{\Delta f_{d}}=10 \mathrm{~Hz}, \delta l=50 \mathrm{~m}$, $\delta x=150 \mathrm{~m}$, and $\delta H=10 \mathrm{~m}$. (a) $\delta v=0.1 \mathrm{~m} / \mathrm{s}$ and (b) $\delta v=10 \mathrm{~m} / \mathrm{s}$


Figure 5.15 GDOP when the distance between satellites $=50 \mathrm{~km}, \sigma_{\Delta t}=80 \mathrm{~ns}, \sigma_{\Delta f_{d}}=10 \mathrm{~Hz}, \delta l=50 \mathrm{~m}$, $\delta x=150 \mathrm{~m}$, and $\delta v=1 \mathrm{~m} / \mathrm{s}$. (a) $\delta H=100 \mathrm{~m}$ and (b) $\delta H=1000 \mathrm{~m}$
so the inclination $i$ and eccentricity $e$ of the actual geosynchronous orbit cannot absolutely be zero, but vary regularly around a small value.
The perturbation of two satellites is different, which makes the radial velocity component of two satellites relative to the transmitting station on earth different most of the time. Thus, different Doppler shifts are caused on the main lobe and side lobe signals of the emitter and retransmitted signals by satellites. The doppler shift will be retransmitted along with the signals and little FDOA will be embodied in the frequency of two signals received by the ground station. An equal-FDOA surface can be determined in 3D space according to FDOA and the location of the emitter can be determined according to an equal-FDOA surface and an equal-TDOA surface $[6,7]$.


Figure 5.16 Diagram of the principle of HEO dual-satellite TDOA-FDOA geolocation

Because the radial component of satellite velocity is different and the velocity component varies with the variation of satellite perturbation, the FDOA line varies continuously all day during a period of 12 hours. In the actual geolocation system, if the geolocation error is within 100 km , the accuracy of the FDOA estimation must be at the level of about 10 mHz . Because the frequency resolution is determined by data length, difference of channel features, and local oscillator (LO) drift of the satellite transmitter, the influence on the estimation of FDOA, makes the level of accuracy difficult to reach in actual estimations of FDOA. In addition, inaccuracy of the satellite ephemeris directly increases the geolocation error. In order to reduce these geolocation errors, a special calibration process must be adopted before TDOA-FDOA geolocation takes place.

### 5.4.2 Calibration Method Based on Reference Sources

The propagation error of the satellite signal, the satellite ephemeris error, and the error caused by satellite-borne equipment will greatly affect the geolocation error. The propagation error and the delay of satellite-borne equipment will influence TDOA of signals. Error caused by LO drift of a satellite transmitter will directly affect the measurement of FDOA. Error of a satellite location will directly affect the TDOA and error of satellite velocity will directly affect the FDOA of signals. In order to decrease the influences of such errors, reference [6] suggests using the auxiliary method of the reference signal to localize.
Suppose a signal that is applicable to the corresponding frequency band and polarization of the two satellites can be received when receiving signals from the emitter. The propagation path after they enter the receiving antenna of the satellite is the same, so satellite delay, error of the transmitter, and the LO frequency error can mostly be eliminated. In addition, the reference signal is transmitted from a known location, which correlates the geolocation of the emitter with the location of the reference signal. Therefore, the influence of the error of the satellite location and velocity is decreased. The reference signal can also be used to improve geolocation accuracy as well as improving the accuracy of parameter estimation.


Figure 5.17 Sketch of the geolocation method based on the reference station

As shown in Figure 5.17 , suppose $\mathbf{r}, \mathbf{r}_{0}$, and $\mathbf{r}_{m}$ are the vectors of the emitter, the reference station and the ground station in the ECEF coordinates system, respectively, $c$ is the velocity of light, $f^{r}$ and $f^{u}$ are the frequency of reference signal and emitter signal, $l_{21}(\mathbf{r})=\left|\mathbf{l}_{2}^{u}-\mathbf{l}_{1}^{u}\right|$ and $l_{21}\left(\mathbf{r}_{0}\right)=\left|\mathbf{l}_{2}^{r}-\mathbf{l}_{1}^{r}\right|$ are the range differences of reference signals to two satellites, $v_{1}(\mathbf{r}), v_{2}(\mathbf{r})$, and $v_{1}\left(\mathbf{r}_{0}\right), v_{2}\left(\mathbf{r}_{0}\right)$ are the components of the velocity vectors $\mathbf{v}_{1}$ and $\mathbf{v}_{2}$ of two satellites in the radial directions of the emitter and reference station, and $v_{1}\left(\mathbf{r}_{m}\right)$ and $v_{2}\left(\mathbf{r}_{m}\right)$ are the components of the velocity vectors $\mathbf{v}_{1}$ and $\mathbf{v}_{2}$ of two satellites in the radial directions of ground station:

$$
\left.\begin{array}{l}
v_{1}\left(\mathbf{r}_{m}\right)=\mathbf{v}_{1}^{\mathrm{T}} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{m}\right)  \tag{5.46}\\
v_{2}\left(\mathbf{r}_{m}\right)=\mathbf{v}_{2}^{\mathrm{T}} \cdot \mathbf{u}_{2}\left(\mathbf{r}_{m}\right)
\end{array}\right\}
$$

where $\mathbf{u}_{1}\left(\mathbf{r}_{m}\right)$ and $\mathbf{u}_{2}\left(\mathbf{r}_{m}\right)$ are the unit vectors of LOS between the primary/auxiliary satellite and the ground station, respectively:

$$
\left.\begin{array}{l}
\mathbf{u}_{1}\left(\mathbf{r}_{m}\right)=\frac{\mathbf{1}_{1}^{r}}{\left|\mathbf{r}_{1}^{r}\right|}  \tag{5.47}\\
\mathbf{u}_{2}\left(\mathbf{r}_{m}\right)=\frac{\mathbf{I}_{2}^{r}}{\left|\mathbf{I}_{2}^{r}\right|}
\end{array}\right\} .
$$

According to the geometric relationship described in Figure 5.17, TDOAs of the emitter signal and the reference signal can be obtained as

$$
\left.\begin{array}{rl}
T D O A_{u} & =\frac{1}{c}\left(l_{2}^{u}+l_{2}-l_{1}^{u}-l_{1}\right)  \tag{5.48}\\
T D O A_{r} & =\frac{1}{c}\left(l_{2}^{r}+l_{2}-l_{1}^{r}-l_{1}\right)
\end{array}\right\}
$$

Perform the subtraction between TDOA of the emitter signal and TDOA of the reference signal, which yields

$$
\begin{aligned}
\operatorname{TDOA}_{u}-\text { TDOA }_{r} & =\frac{1}{c}\left(l_{2}^{u}+l_{2}-l_{1}^{u}-l_{1}\right)-\frac{1}{c}\left(l_{2}^{r}+l_{2}-l_{1}^{r}-l_{1}\right) \\
& =\frac{1}{c}\left(l_{2}^{u}-l_{1}^{u}\right)-\frac{1}{c}\left(l_{2}^{r}-l_{1}^{r}\right)
\end{aligned}
$$

$$
\begin{equation*}
=\frac{1}{c}\left[l_{21}(\mathbf{r})-l_{21}\left(\mathbf{r}_{0}\right)\right] . \tag{5.49}
\end{equation*}
$$

FDOAs of the emitter signal and reference signal are

$$
\left.\begin{array}{l}
F D O A_{r}=\frac{f^{r}}{c} v_{1}\left(\mathbf{r}_{0}\right)+\frac{\left(f^{r}-f^{\mathrm{T}}\right)}{c} v_{1}\left(\mathbf{r}_{m}\right)-\frac{f^{r}}{c} v_{2}\left(\mathbf{r}_{0}\right)-\frac{\left(f^{r}-f^{\mathrm{T}}\right)}{c} v_{2}\left(\mathbf{r}_{m}\right)  \tag{5.50}\\
F D O A_{u}=\frac{f^{u}}{c} v_{1}(\mathbf{r})+\frac{\left(f^{u}-f^{\mathrm{T}}\right)}{c} v_{1}\left(\mathbf{r}_{m}\right)-\frac{f^{u}}{c} v_{2}(\mathbf{r})-\frac{\left(f^{u}-f^{\mathrm{T}}\right)}{c} v_{2}\left(\mathbf{r}_{m}\right)
\end{array}\right\}
$$

where $f^{T}$ is the local retransmitting frequency of the transceiver of the HEO satellite. Perform the subtraction between FDOA of the emitter signal and FDOA of the reference signal, which yields

$$
\begin{align*}
F D O A_{u}-F D O A_{r}= & \frac{f^{u}}{c} v_{1}(\mathbf{r})-\frac{f^{r}}{c} v_{1}\left(\mathbf{r}_{0}\right)-\frac{f^{u}}{c} v_{2}(\mathbf{r})+\frac{f^{r}}{c} v_{2}\left(\mathbf{r}_{0}\right)-\frac{f^{u}-f^{r}}{c}\left[v_{2}\left(\mathbf{r}_{m}\right)-v_{2}\left(\mathbf{r}_{m}\right)\right] \\
= & -\frac{f^{u}}{c}\left[v_{2}(\mathbf{r})-v_{1}(\mathbf{r})\right]+\frac{f^{u}}{c}\left[v_{2}\left(\mathbf{r}_{0}\right)-v_{1}\left(\mathbf{r}_{0}\right)\right]-\frac{f^{u}-f^{r}}{c}\left[v_{2}\left(\mathbf{r}_{0}\right)-v_{1}\left(\mathbf{r}_{0}\right)\right] \\
& -\frac{f^{u}-f^{r}}{c}\left[v_{2}\left(\mathbf{r}_{m}\right)-v_{2}\left(\mathbf{r}_{m}\right)\right] . \tag{5.51}
\end{align*}
$$

Let $v_{21}(\mathbf{r})$ and $v_{21}\left(\mathbf{r}_{0}\right)$ be the components of the satellite velocity vector difference in the radial direction of the transmitting station and the reference station, that is,

$$
\left.\begin{array}{l}
v_{21}(\mathbf{r})=v_{2}(\mathbf{r})-v_{1}(\mathbf{r})  \tag{5.52}\\
v_{21}\left(\mathbf{r}_{0}\right)=v_{2}\left(\mathbf{r}_{0}\right)-v_{1}\left(\mathbf{r}_{0}\right) \\
v_{21}\left(\mathbf{r}_{m}\right)=v_{2}\left(\mathbf{r}_{m}\right)-v_{1}\left(\mathbf{r}_{m}\right)
\end{array}\right\}
$$

The above expressions can be simplified as

$$
\begin{equation*}
F D O A_{u}-F D O A_{r}=-\frac{f^{u}}{c}\left[v_{21}(\mathbf{r})-v_{21}\left(\mathbf{r}_{0}\right)\right]-\frac{f^{u}-f^{r}}{c}\left[v_{21}\left(\mathbf{r}_{0}\right)+v_{21}\left(\mathbf{r}_{m}\right)\right] . \tag{5.53}
\end{equation*}
$$

Combine with the earth's surface expression and suppose that the earth's surface is a regular spherical surface:

$$
\begin{equation*}
|\mathbf{r}|=R, \tag{5.54}
\end{equation*}
$$

where $R$ is the radius of the earth. Combine expressions (5.49), (5.53), and (5.54), and then solve to obtain the geolocation solution.
Because the TDOA expression and the FDOA expression are nonlinear functions of the emitter position, and it is difficult to obtain an effective analytic solution, the Taylor series method [8] can be used. Perform Taylor's expansion of $l_{21}(\mathbf{r})$ and $v_{21}(\mathbf{r})$ in geolocation expressions at the reference station $\mathbf{r}_{0}$. Then, the following will be obtained:

$$
\left.\begin{array}{l}
l_{21}(\mathbf{r})=l_{21}\left(\mathbf{r}_{0}\right)+\left(\mathbf{r}-\mathbf{r}_{0}\right)^{\mathrm{T}} \nabla l_{21}\left(\mathbf{r}_{0}\right)+\text { HOT } \\
v_{21}(\mathbf{r})=v_{21}\left(\mathbf{r}_{0}\right)+\left(\mathbf{r}-\mathbf{r}_{0}\right)^{\mathrm{T}} \nabla v_{21}\left(\mathbf{r}_{0}\right)+\text { HOT }
\end{array}\right\}
$$

Omit the higher order terms (HOTs) and substitution into the geolocation expression yields

$$
\left.\begin{array}{l}
\mathbf{r}^{\mathrm{T}} \nabla l_{21}\left(\mathbf{r}_{0}\right)=R\left|\nabla l_{21}\left(\mathbf{r}_{0}\right)\right| \cos \gamma \approx \Delta K_{21}^{l}\left(\mathbf{r}, \mathbf{r}_{0}\right)+\mathbf{r}_{0}^{\mathrm{T}} \nabla l_{21}\left(\mathbf{r}_{0}\right)  \tag{5.55}\\
\mathbf{r}^{\mathrm{T}} \nabla v_{21}\left(\mathbf{r}_{0}\right)=R\left|\nabla v_{21}\left(\mathbf{r}_{0}\right)\right| \sin \delta \approx \Delta K_{21}^{v}\left(\mathbf{r}, \mathbf{r}_{0}\right)+\mathbf{r}_{0}^{\mathrm{T}} \nabla v_{21}\left(\mathbf{r}_{0}\right)
\end{array}\right\}
$$

where $\gamma$ is the included angle between the vector $\nabla l_{21}\left(\mathbf{r}_{0}\right)$ and the emitter position vector $\mathbf{r}$, $\delta$ is the included angle between the vector $\nabla v_{21}\left(\mathbf{r}_{0}\right)$ and the emitter position vector $\mathbf{r}, \nabla=$ $[\partial / \partial x \partial / \partial y \partial / \partial z]$ is the derivative vector of measurement, and:

$$
\begin{aligned}
& \Delta K_{21}^{l}\left(\mathbf{r}, \mathbf{r}_{0}\right)=l_{21}(\mathbf{r})-l_{21}\left(\mathbf{r}_{0}\right)=c\left[T D O A_{u}-\text { TDOA }_{r}\right], \\
& \Delta K_{21}^{l}\left(\mathbf{r}, \mathbf{r}_{0}\right)=v_{21}(\mathbf{r})-v_{21}\left(\mathbf{r}_{0}\right)=-\frac{c}{f^{u}}\left[F D O A_{u}-F D O A_{r}\right]-\frac{f^{u}-f^{r}}{c}\left[v_{21}\left(\mathbf{r}_{0}\right)+v_{21}\left(\mathbf{r}_{m}\right)\right] .
\end{aligned}
$$

By analyzing expression (5.55) we know that the values of the items on both sides of the expression depend on the measured values of TDOA and FDOA, the location and velocity of the primary satellite and the auxiliary satellite, the location of the reference station and the ground station, and that these conditions and data can be measured. The gradient vector on the left side of the geolocation expressions also depends on the location and velocity of the primary satellite and the auxiliary satellite as well as the location of the reference station. Therefore, in the geolocation equations, only the angle of $\gamma$ and $\delta$ are unknown and can to be solved according to expression (5.55). Suppose the earth is a regular spheroid; establish a coordinate system with the auxiliary satellite as the center and the two orthogonal vectors $\nabla l_{21}\left(\mathbf{r}_{0}\right)$ and $\nabla v_{21}\left(\mathbf{r}_{0}\right)$ as the $x$ and $y$ axes. Use the radials forming the two angles of $\gamma$ and $\delta$ to intersect the spherical surface of the earth and then use the method described in Section 3.2 to determine the vector $\mathbf{r}$ of the target emitter on ECEF coordinates, so as to determine the true location of the emitter according to the earth's surface.
The extended linearization solution of the above equations is based on the Taylor expanding method. The precondition for such a solution is that $\mathbf{r}$ is located within the adjacent area of $\mathbf{r}_{0}$, so that HOTs can be omitted without causing great error. If the coordinates of the target transmitting station is not known, it is difficult to find a reference station that is close to the true emitter. Thus, errors are introduced during the actual extended linearization solution of the equations. In order to decrease such errors, the iteration method can be used to improve geolocation accuracy.
Firstly, use the vector $\mathbf{r}_{0}$ of the initial location of the reference station to solve expression (5.55) in order to obtain the geolocation solution $\mathbf{r}_{1}$. Then substitute $\mathbf{r}_{1}$ for $\mathbf{r}_{0}$ as the reference station for the new solution and equations will then become

$$
\left.\begin{array}{r}
\mathbf{r}^{\mathrm{T}} \nabla l_{21}\left(\mathbf{r}_{1}\right)=R\left|\nabla l_{21}\left(\mathbf{r}_{0}\right)\right| \cos \gamma \approx \Delta K_{21}^{l}\left(\mathbf{r}, \mathbf{r}_{1}\right)+\mathbf{r}_{1}^{\mathrm{T}} \nabla l_{21}\left(\mathbf{r}_{1}\right) \\
\mathbf{r}^{\mathrm{T}} \nabla v_{21}\left(\mathbf{r}_{1}\right)=R\left|\nabla v_{21}\left(\mathbf{r}_{1}\right)\right| \sin \delta \approx \Delta K_{21}^{v}\left(\mathbf{r}, \mathbf{r}_{1}\right)+\mathbf{r}_{1}^{\mathrm{T}} \nabla v_{21}\left(\mathbf{r}_{1}\right)
\end{array}\right\},
$$

where $\Delta K_{21}^{l}\left(\mathbf{r}, \mathbf{r}_{1}\right)$ and $\Delta K_{21}^{v}\left(\mathbf{r}, \mathbf{r}_{1}\right)$ are iterated according to the following expression:

$$
\left.\begin{array}{l}
\Delta K_{21}^{l}\left(\mathbf{r}, \mathbf{r}_{1}\right)=\Delta K_{21}^{l}\left(\mathbf{r}, \mathbf{r}_{0}\right)-\left[l_{21}\left(\mathbf{r}_{1}\right)-l_{21}\left(\mathbf{r}_{0}\right)\right] \\
\Delta K_{21}^{v}\left(\mathbf{r}, \mathbf{r}_{1}\right)=\Delta K_{21}^{v}\left(\mathbf{r}, \mathbf{r}_{0}\right)-\left[v_{21}\left(\mathbf{r}_{1}\right)-v_{21}\left(\mathbf{r}_{0}\right)\right]
\end{array}\right\} .
$$

Solve these equations again to obtain a new geolocation solution $\mathbf{r}_{2}$, which can be used as the reference station of the new round of solutions. With the continuous renewal of locations of such equivalent reference stations, the geolocation calculation enters into an iterative process. In general, the reference station is not close to the emitter, so the solution of expressions will be much closer to the true location of the emitter than the reference station. When the equivalent reference station is gradually close to the true emitter, the fluctuation of the geolocation error
becomes smaller. Therefore, iterative times can be set to end such iterations or the difference between two geolocations can be set as the converged condition of stopping iteration:

$$
\left|\mathbf{r}_{j}-\mathbf{r}_{i}\right|<\varepsilon
$$

Tests show that when parametric errors in equations are small, the convergence of such iterative processes is good. Generally a geolocation accuracy of dozens of kilometers can be achieved through three to five iterations.

### 5.4.3 Calibration Method Using Multiple Reference Sources

In the previous analysis, the error of satellite ephemeris is not taken into consideration or the influences caused by satellite ephemeris are considered to have been omitted. However, in the actual geolocation process, this assumption cannot hold. For example, for a signal in the Ku-band of 14 GHz , if the velocity error of satellite motion is $0.02 \mathrm{~mm} / \mathrm{s}$, a 1 mHz Doppler frequency error will be caused and the resulting geolocation error will be 1 km . The actual radial velocity error of the satellite can be up to $0.2 \mathrm{~m} / \mathrm{s}$, which will cause a geolocation error of thousands of kilometers, so a proper handling method must be adopted.
Therefore, reference [6] suggests taking the measurement results of multiple reference stations as references and calibrating the ephemeris error to achieve a better effect.

### 5.4.3.1 Theory of Multiple Reference Stations Calibration Geolocation

In order to analyze and derive the calibration method using a multiple reference station, let us start with the analysis of the following system of geolocation expressions:

$$
\left.\begin{array}{l}
c\left[\mathrm{TDOA}_{u}-\mathrm{TDOA}_{r}\right]=l_{21}(\mathbf{r})-l_{21}\left(\mathbf{r}_{0}\right)  \tag{5.56}\\
\frac{-c}{f^{u}}\left[\mathrm{FDOA}_{u}-\mathrm{FDOA}_{r}\right]=v_{21}(\mathbf{r})-v_{21}\left(\mathbf{r}_{0}\right)+\frac{f^{u}-f^{r}}{f^{u}}\left[v_{21}\left(\mathbf{r}_{0}\right)+v_{21}\left(\mathbf{r}_{m}\right)\right]
\end{array}\right\} .
$$

Firstly, consider the influence of the satellite velocity error on geolocation. Because $v_{21}(\mathbf{r})=$ $\mathbf{v}_{2} \cdot \mathbf{u}_{2}(\mathbf{r})-\mathbf{v}_{1} \cdot \mathbf{u}_{1}(\mathbf{r})$, the above FDOA expression can be rewritten as

$$
\begin{align*}
\frac{-c}{f^{u}}\left[\mathrm{FDOA}_{u}-\mathrm{FDOA}_{r}\right]= & v_{21}(\mathbf{r})-\frac{f^{r}}{f^{u}} v_{21}\left(\mathbf{r}_{0}\right)+\frac{f^{u}-f^{r}}{f^{u}} v_{21}\left(\mathbf{r}_{m}\right) \\
= & \mathbf{v}_{2} \cdot \mathbf{u}_{2}(\mathbf{r})-\mathbf{v}_{1} \cdot \mathbf{u}_{1}(\mathbf{r})-\frac{f^{r}}{f^{u}}\left[\mathbf{v}_{2} \cdot \mathbf{u}_{2}\left(\mathbf{r}_{0}\right)-\mathbf{v}_{1} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{0}\right)\right] . \\
& +\frac{f^{u}-f^{r}}{f^{u}}\left[\mathbf{v}_{2} \cdot \mathbf{u}_{2}\left(\mathbf{r}_{m}\right)-\mathbf{v}_{1} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{m}\right)\right] \tag{5.57}
\end{align*}
$$

Fundamentally, FDOA caused by satellite velocity is the effect on the corresponding radial component generated by it only, and this effect is a scalar. In other words, if the quantity equivalent to the scalar can be determined, it can be used for the final geolocation, while the accurate velocities of two satellites are not required to be determined. Because FDOA is caused by the relative velocity between two satellites, if the accurate velocity of two satellites is $\mathbf{v}_{1}$ and $\mathbf{v}_{2}$, the same FDOA effect can be generated by using satellite velocity $\mathbf{v}_{1}^{\prime}$ and $\mathbf{v}_{2}^{\prime}$ with error and
a velocity increment $\Delta \mathbf{v}$ between the two satellite velocities, that is, the following expression will be obtained according to the FDOA expression (5.56):

$$
\begin{align*}
\mathbf{v}_{2} \cdot \mathbf{u}_{2}(\mathbf{r})- & \mathbf{v}_{1} \cdot \mathbf{u}_{1}(\mathbf{r})-\frac{f^{r}}{f^{u}}\left[\mathbf{v}_{2} \cdot \mathbf{u}_{2}\left(\mathbf{r}_{0}\right)-\mathbf{v}_{1} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{0}\right)\right] \\
& +\frac{f^{u}-f^{r}}{f^{u}}\left[\mathbf{v}_{2} \cdot \mathbf{u}_{2}\left(\mathbf{r}_{\mathrm{m}}\right)-\mathbf{v}_{1} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{\mathrm{m}}\right)\right] \\
= & \left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}(\mathbf{r})-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}(\mathbf{r})-\frac{f^{r}}{f^{u}}\left[\left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}\left(\mathbf{r}_{0}\right)-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{0}\right)\right] . \\
& +\frac{f^{u}-f^{r}}{f^{u}}\left[\left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}\left(\mathbf{r}_{\mathrm{m}}\right)-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{\mathrm{m}}\right)\right] \tag{5.58}
\end{align*}
$$

Because $\Delta \mathbf{v}$ is a vector in 3D space with three unknowns, it can certainly be applicable to expression (5.58) for at least the following system of expressions has positive definite solutions:

$$
\left.\begin{array}{l}
\mathbf{v}_{2} \cdot \mathbf{u}_{2}(\mathbf{r})-\mathbf{v}_{1} \cdot \mathbf{u}_{1}(\mathbf{r})=\left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}(\mathbf{r})-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}(\mathbf{r})  \tag{5.59}\\
\mathbf{v}_{2} \cdot \mathbf{u}_{2}\left(\mathbf{r}_{0}\right)-\mathbf{v}_{1} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{0}\right)=\left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}\left(\mathbf{r}_{0}\right)-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{0}\right) \\
\mathbf{v}_{2} \cdot \mathbf{u}_{2}\left(\mathbf{r}_{m}\right)-\mathbf{v}_{1} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{m}\right)=\left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}\left(\mathbf{r}_{m}\right)-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{m}\right)
\end{array}\right\} .
$$

In fact, solution of expression (5.59) is only required to satisfy the special cases of conditional infinite $\Delta \mathbf{v}$, because expression (5.58) is a scalar expression. Only the comprehensive scalars of the expression are required to be equal, while three subitems are not required to be equal. Accurate velocity is not required for the right side of expression (5.58), so in the actual analysis, ephemeris with error can be obtained by directly using ephemeris forecast or via other means.
From expression (5.58) we know that despite the influence on $\mathbf{r}$ (the radial direction of the emitter), the velocity increment $\Delta \mathbf{v}$ also has an influence on $\mathbf{r}_{0}$ (the radial direction of the reference station). Because the position of the reference station is accurate and known, if the equivalent $\Delta \mathbf{v}$ is reversely analyzed by using the geolocation bias of the reference station caused by several $\Delta \mathbf{v}$, such influences can be eliminated by localizing the emitter. This is the method for ephemeris calibration using a multiple reference station.

### 5.4.3.2 Four-Station Calibration Method

In order to determine the sole $\Delta \mathbf{v}$ to correct the influence of the satellite velocities $\mathbf{v}_{1}^{\prime}$ and $\mathbf{v}_{2}^{\prime}$ on geolocation, supposing there are $M$ reference stations, which is similar to expression (5.58), we can obtain $M$ systems of expressions [6]:

$$
\left.\begin{array}{rl}
c\left[\mathrm{TDOA}_{u}-\mathrm{TDOA}_{r_{i}}\right]= & l_{21}(\mathbf{r})-l_{21}\left(\mathbf{r}_{i}\right) \quad(i=1, \ldots, M) \\
\frac{-c}{f^{u}}\left[\mathrm{FDOA}_{u}-\mathrm{FDOA}_{r_{i}}\right]= & \left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}(\mathbf{r})-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}(\mathbf{r}) \\
& -\frac{f^{r_{i}}}{f^{u}}\left[\left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}\left(\mathbf{r}_{i}\right)-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{i}\right)\right]  \tag{5.60}\\
& +\frac{f^{u}-f^{r_{i}}}{f^{u}}\left[\left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}\left(\mathbf{r}_{m}\right)-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{m}\right)\right] \\
|\mathbf{r}|= & R
\end{array}\right\} .
$$

In expression (5.60), only $\Delta \mathbf{v}$ and the emitter $\mathbf{r}$ are unknown. $\Delta \mathbf{v}$ will not exceed the possible maximum value of perturbation velocity, that is, $\Delta \mathbf{v}$ varies only in a certain velocity range because the range of satellite perturbation is relatively fixed. If a value of $\Delta \mathbf{v}$ is taken from this range, the system of expression (5.60) can be solved to obtain $\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)$. If the value of $\Delta \mathbf{v}$ cannot correct the influence of satellite velocity $\mathbf{v}_{1}^{\prime}$ and $\mathbf{v}_{2}^{\prime}$ on geolocation, $\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)$ will deviate from the true emitter location r. If the value of $\Delta \mathbf{v}$ can be adjusted so that it can correct the influence of the satellite velocities $\mathbf{v}_{1}^{\prime}$ and $\mathbf{v}_{2}^{\prime}$ with error on geolocation, $\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)$ is the emitter location $\mathbf{r}$. Thus, accurate geolocation is achieved.
Certainly, $\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)-\mathbf{r}$ can be used to perform theoretic analysis, but in reality the emitter location $\mathbf{r}$ is unknown and cannot be used for the determination of $\Delta \mathbf{v}$. Therefore, other measurement factors must be used. For each reference station, a solution can be obtained by solving expression (5.60), and such solutions are relatively accurate and close to the true emitter location $\mathbf{r}$. Therefore, whether the selected value of $\Delta \mathbf{v}$ is optimal can be judged by the degree of concentration of such solutions [6]. The simple function reflecting the degree of concentration is the sum of the Euclidean distance between two solutions, that is,

$$
\begin{equation*}
J=\sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M}\left|\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)-\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{j}\right)\right| \tag{5.61}
\end{equation*}
$$

As $\Delta \mathbf{v}$ is a 3 D vector, the process of adjusting its value in fact is to minimize the error of the 3D cost function, $M \geq 4$. Figure 5.18 is the sketch of the four-reference station ephemeris calibration.


Figure 5.18 Sketch of four-reference station ephemeris calibration

In the above ephemeris calibration process, only when the TDOA-FDOA parameter, the carrier frequency of the reference signal and the emitter signal, and the coordinates of the receiving station and the reference station have no error, can optimal results be achieved for the whole process, but in actual application that is not possible. The error of such factors makes unpredictable local minima exist in the process of iterative search. If the ideal iterative process is compared to a process continuously heading to the 'bowl bottom' representing the minimum error, the aforesaid error will bring about more bumpy 'traps' of the 'bowl' rather than making it smoothly slide downwards. The local minima will not generate effective results from the iterative searching process for determining actual $\Delta \mathbf{v}$, but will cause incorrect calibration.
Generally speaking, more samples can be provided by adding the number of reference stations. However, adding samples alone can only decrease the probability of local minima, but cannot essentially eliminate such influences. Thus, the direct criterion for judging local optimum and global optimum cannot be provided; in other words, the cost function can only provide a local optimal result instead of a global optimal result, as shown in the following expression:

$$
\begin{align*}
J & =\sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M}\left|\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)-\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{j}\right)\right| \\
& =\sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M}\left|\left[\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)-\mathbf{r}\right]-\left[\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{j}\right)-\mathbf{r}\right]\right| \neq \sum_{i=1}^{M}\left|\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)-\mathbf{r}\right| . \tag{5.62}
\end{align*}
$$

### 5.4.3.3 Five-Station Calibration Geolocation Algorithm

In expressions (5.56) to (5.61), what we considered was using reference stations to complete ephemeris calibration and the unknown location of the emitter. If five reference sources are provided, we can separate the two parts of the work. Firstly, only ephemeris calibration is considered. Substitute a reference station with a known location for the original emitter, adjust $\Delta \mathbf{v}$ and compare with the location of the reference station, and provide a direct judge criterion of the local optimum and global optimum, in order to complete the ephemeris calibration. Finally, use this ephemeris value to complete geolocation, so as to obtain improved geolocation accuracy [6].
After substituting the reference station $\mathbf{r}_{0}$ for the unknown emitter, the system of geolocation expressions is turned to become

$$
\left.\begin{array}{rl}
c\left[\mathrm{TDOA}_{r_{0}}-\mathrm{TDOA}_{r_{i}}\right]= & l_{21}\left(\mathbf{r}_{0}\right)-l_{21}\left(\mathbf{r}_{i}\right) \quad(i=1, \ldots, M) \\
\frac{-c}{f^{r_{0}}}\left[\mathrm{FDOA}_{r_{0}}-\mathrm{FDOA}_{r_{i}}\right]= & \left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}\left(\mathbf{r}_{0}\right)-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{0}\right) \\
& -\frac{f^{r_{i}}}{f^{r_{0}}}\left[\left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}\left(\mathbf{r}_{\mathrm{i}}\right)-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{i}\right)\right]  \tag{5.63}\\
& +\frac{f^{r_{0}}-f^{r_{i}}}{f^{r_{0}}}\left[\left(\mathbf{v}_{2}^{\prime}+\Delta \mathbf{v}\right) \cdot \mathbf{u}_{2}\left(\mathbf{r}_{m}\right)-\mathbf{v}_{1}^{\prime} \cdot \mathbf{u}_{1}\left(\mathbf{r}_{m}\right)\right] \\
\left|\mathbf{r}_{0}\right|= & R
\end{array}\right\}
$$

Solve the above system of geolocation expressions to obtain the geolocation solution $\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)$ of reference station $\mathbf{r}_{0}$. Generally, there is a certain degree of error between the geolocation solution $\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)$ and the true location due to the influence of the parameter estimation error. Now we can directly define the error as $\varepsilon=\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)-\mathbf{r}_{0}$, in order to realizecalibration processing by defining the following cost function:

$$
\begin{equation*}
J=\sum_{i=1}^{M}|\varepsilon|=\sum_{i=1}^{M}\left|\mathbf{r}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)-\mathbf{r}_{0}\right| \tag{5.64}
\end{equation*}
$$

Certainly, when defining the cost function, $|\varepsilon|^{2}$ can be used as the measurement of error, which is equivalent to expression (5.64). No matter what kind of error is adopted, remember that in iterative processing the emitter to be localized is on the earth's surface. Therefore, it is least squares (LS) processing with a spherical constraint condition. If this fact is ignored, unnecessary error will be caused. Compared with Figure 5.18, Figure 5.19 is a sketch of the ephemeris calibration by five known reference stations (including $\mathbf{r}_{0}$ ).
Compared with the ephemeris calibration method, which directly localizes the emitter (Section 5.4.3.2), accurate coordinates of $\mathbf{r}_{0}$ can ensure that an iterative search will not be caught in local optimization, accuracy of $f^{r_{0}}$ is known, and its error is smaller than that of $f^{u}$ obtained through measurement. In addition, even though $M+1$ reference stations are used in Figure 5.19, the locating point $\mathbf{r}_{0}\left(\Delta \mathbf{v}, \mathbf{r}_{i}\right)$ is not used to establish the cost function; in fact only $M$ reference stations are needed. Any one of these reference stations can be used as the


Figure 5.19 Sketch of five-station ephemeris calibration
reference station of the emitter, to provide a calibration result. Thus, the fusion of several results can be performed and better geolocation accuracy can be obtained [6].
Use the similar analytical method to calibrate the error of satellite location. Because $l_{21}(\mathbf{r})=$ $\left|\mathbf{r}_{2}-\mathbf{r}\right|-\left|\mathbf{r}_{1}-\mathbf{r}\right|$, the TDOA expression of (5.56) can be turned into

$$
\begin{align*}
c\left[\mathrm{TDOA}_{u}-T D O A_{r}\right] & =\left|\mathbf{r}_{2}-\mathbf{r}\right|-\left|\mathbf{r}_{1}-\mathbf{r}\right|-\left(\left|\mathbf{r}_{2}-\mathbf{r}_{0}\right|-\left|\mathbf{r}_{1}-\mathbf{r}_{0}\right|\right) \\
& =\left|\mathbf{r}_{2}^{\prime}+\Delta \mathbf{r}_{s}-\mathbf{r}\right|-\left|\mathbf{r}_{1}^{\prime}-\mathbf{r}\right|-\left(\left|\mathbf{r}_{2}^{\prime}+\Delta \mathbf{r}_{s}-\mathbf{r}\right|-\left|\mathbf{r}_{1}^{\prime}-\mathbf{r}_{0}\right|\right) \tag{5.65}
\end{align*}
$$

This includes the satellite locations, $\mathbf{r}_{1}^{\prime}$ and $\mathbf{r}_{2}^{\prime}$, with error includinging a location increment $\Delta \mathbf{r}_{s}$, which can also be applicable to the TDOA geolocation expression as the true satellite locations, $\mathbf{r}_{1}$ and $\mathbf{r}_{2}$. In order to calibrate the influence of ephemeris and avoid the location optimization indicated in expression (5.62), we directly use $M+1$ reference stations to form the following expressions:

$$
\left.\begin{array}{rl}
c\left[\mathrm{TDOA}_{u}-\mathrm{TDOA}_{r_{i}}\right] & =\left|\mathbf{r}_{2}^{\prime}+\Delta \mathbf{r}_{s}-\mathbf{r}_{0}\right|-\left|\mathbf{r}_{1}^{\prime}-\mathbf{r}_{0}\right|-\left(\left|\mathbf{r}_{2}^{\prime}+\Delta \mathbf{r}_{s}-\mathbf{r}_{i}\right|-\left|\mathbf{r}_{1}^{\prime}-\mathbf{r}_{i}\right|\right) \\
\frac{-c}{f^{u}}\left[\mathrm{FDOA}_{u}-F D O A_{r_{i}}\right] & =v_{21}\left(\mathbf{r}_{0}\right)-\frac{f^{r_{i}}}{f^{r_{0}}} v_{21}\left(\mathbf{r}_{i}\right)+\frac{f^{r_{0}}-f^{r_{i}}}{f^{r_{0}}} v_{21}\left(\mathbf{r}_{m}\right)(i=1, \ldots, M)  \tag{5.66}\\
\left|\mathbf{r}_{0}\right| & =R
\end{array}\right\} .
$$

Supposing the solution of above expressions is $\mathbf{r}_{0}\left(\Delta \mathbf{r}_{s}, \mathbf{r}_{i}\right)$, we still need at least three samples for the proper location increment $\Delta \mathbf{r}_{s}$, which is found by iterative search, that is, when $M \geq 4$, the cost function can be established:

$$
\begin{equation*}
J=\sum_{i=1}^{M}|\varepsilon|=\sum_{i=1}^{M}\left|\mathbf{r}_{0}\left(\Delta \mathbf{r}_{s}, \mathbf{r}_{i}\right)-\mathbf{r}_{0}\right| \tag{5.67}
\end{equation*}
$$

We can know that the conditions required by calibration of the satellite position error are completely the same as those required by calibration of the satellite velocity error, and there is no contradiction between them. Thus, the same condition of the reference station can be used to realize the complete calibration of the satellite's ephemeris. When calibrating one, the other one is supposed to be correct. Therefore, the satellite velocity is usually calibrated first, followed by the satellite location. There remains an error due to the small influence between them. In order to obtain a better calibration result, one way is to add a reference station to increase redundancy for fusion processing; the other way is to continuously calibrating one satellite perturbation period in the time domain and perform time domain fusion to obtain a complete ephemeris calibration function.

### 5.4.4 Flow of Calibration and Geolocation

By summarizing the above ephemeris calibration method, the following steps can be adopted [6]:

1. Select a sufficient number of reference stations, at least four usually, and then acquire signals and estimating parameters. The selection principle of reference station is as follows.
2. Estimate the TDOA-FDOA parameter of all signals.
3. Use expression (5.64) to calibrate the satellite velocity error.
4. Use expression (5.67) to calibrate the satellite position.
5. Perform the necessary calibration fusion or form a calibration function.
6. Carry out the final emitter geolocation.

According to the derivation of the ephemeris calibration method, we know that reference stations must be properly selected in order to realize effective calibration.

1. In order to achieve a good ephemeris calibration result, especially in order to overcome the error influence on the satellite ephemeris, at least five reference stations must be selected so that redundant information can assist in improving calibration accuracy.
2. When using multiple reference stations for ephemeris calibration, how to find roots in 2D space is involved and an ill-conditioned coefficient matrix should be avoided to some extent. For selection of the reference station, it is better to select spatial separated stations rather than two stations in the same place.
3. The optimal case is for reference stations to be distributed around the emitter area, and on that basis the less distribution area there is for these reference stations the better the geolocation accuracy will be. Such a case is for all the reference stations to around the emitter, which is favorable for reducing measurement error.
4. Selecting a reference station close to the emitter for ephemeris calibration is favorable for improving calibration accuracy, because it helps to correct the channel difference of the uplink path of signals. In practice, it is feasible to loosely localize the area of the emitter and then select proper reference stations according to the general area.

We can use the above method to calibrate the ephemeris error of satellites. The satellite ephemeris after calibration can be used for geolocation and it will obviously improve system accuracy. In the actual geolocation, requirement for satellite location accuracy is not very high, but the requirement for relative velocity accuracy of the satellite is high. Thus, the FDOA expression can be used alone to correct satellite velocity. A good improvement effect can be achieved after fitting the FDOA data obtained at different times and reducing the error caused in parameter estimation.

### 5.5 Method of Measuring TDOA and FDOA

### 5.5.1 The Cross-Ambiguity Function

In order to estimate and obtain the accurate TDOA and FDOA between two signals, Stein [9] proposed a method based on the cross-ambiguity function (CAF) for a TDOA-FDOA joint estimation. Under the condition of a high SNR (signal-to-noise ratio) (when the SNR of the input signal is higher than 10 dB ), the joint estimation of the time delay and frequency shift based on the CAF is effective, unbiased, and close to the CRLB (Cramér-Rao lower bound).
For any signal $r_{1}(t)$ and $r_{2}(t)$, the CAF denoted as $A(\tau, v)$ can be defined as [9]

$$
\begin{equation*}
\operatorname{CAF}(\tau, v)=\int_{0}^{T} r_{1}(t) r_{2}^{*}(t+\tau) \mathrm{e}^{-j 2 \pi v t} \mathrm{~d} t \tag{5.68}
\end{equation*}
$$



Figure 5.20 Cross-ambiguity function for the BPSK modulated signal

In dual-satellite geolocation, the noise in $r_{1}(t)$ and $r_{2}(t)$ are not correlated while the signal in $r_{1}(t)$ and $r_{2}(t)$ are correlated because they are from the same emitter. If the same signal is received by the receivers of two satellites, when TDOA and FDOA of signal $r_{1}(t)$ and signal $r_{2}(t)$, respectively, are equal to $\tau$ and $v$ after the time-domain accumulation for a period of $T$, the peak value will appear on the surface form by the 2D matrix of $\operatorname{CAF}(\tau, v)$. For example, if the symbol rate of a BPSK (binary phase-shift keying) modulated signal is $1 \mathrm{MBaud} / \mathrm{s}$ and the duration is 10 ms , the $\mathrm{SNR}=-10 \mathrm{~dB}$. The graph of the CAF is shown as in Figure 5.20.
Because measurement accuracy of TDOA and FDOA depends on the accumulated time $T$ and the signal bandwidth $B$, in order to obtain measurement results of high accuracy, enough signal time of integration must have accumulated to perform the CAF calculation. However, as the computation load is very large, in order to decrease the computation load to satisfy real-time processing requirements, the sampling point with a duration of $T$ can on average be divided into $K$ segments, each with a duration of $T / K$. Calculate the ambiguity function $\operatorname{CAF}(\tau, v, k)(k=1,2, \ldots, K)$ on such time intervals and then correlate and plus them to obtain a total cross-ambiguity function $C A F_{C}(\tau, v)$. Perform a 2D search in the CAF matrix to find the TDOA and FDOA corresponding to the peak value.

### 5.5.2 Theoretical Analysis on the TDOA-FDOA Measurement Performance

### 5.5.2.1 FDOA and TDOA Measurement Accuracy

For each given time delay $\tau$ and frequency shift $v$, the calculation results of the CAF include the product of signals and noises. In order to accurately obtain the required peak position, the SNR of input signals will be higher than 10 dB . Under such a restriction, the estimation error comes from noise accumulated around the true peak value. This shows that the CAF method for TDOA and FDOA estimation is unbiased and its variance can reach the CRLB.

Stein [9] provides the CRLB of TDOA-FDOA as

$$
\left.\begin{array}{l}
\sigma_{\mathrm{TDOA}}=\frac{0.55}{B} \frac{1}{\sqrt{B_{n} T \gamma}}  \tag{5.69}\\
\sigma_{\mathrm{FDOA}}=\frac{0.55}{T} \frac{1}{\sqrt{B_{n} T \gamma}}
\end{array}\right\},
$$

where $B$ is the signal bandwidth, $B_{n}$ is the input noise bandwidth, $T$ is the accumulated time of signals, and $\gamma$ is the equivalent input SNR of input signals:

$$
\begin{equation*}
\frac{1}{\gamma}=\frac{1}{2}\left[\frac{1}{\gamma_{1}}+\frac{1}{\gamma_{2}}+\frac{1}{\gamma_{1} \gamma_{2}}\right], \tag{5.70}
\end{equation*}
$$

where $\gamma_{1}$ and $\gamma_{2}$ are the input SNRs of the two receivers. If $\gamma_{1}$ and $\gamma_{2}$ are far higher than 0 dB , the third item in the above expression can be ignored. If $\gamma_{1}=\gamma_{2}$, then $\gamma=\gamma_{1}=\gamma_{2}$ can be obtained. If $\gamma_{1} \ll \gamma_{2}$ and $\gamma_{2} \gg 1$, then $\gamma \approx 2 \gamma_{1}$. If $\gamma_{1}$ and $\gamma_{2}$ are much lower than 0 dB , the equivalent input SNR $\gamma$ can be approximately calculated as $\gamma=2 \gamma_{1} \gamma_{2}$.
Remark. The SNR in the above two expressions is inversely proportional to the bandwidth $B_{n}$ (for wider bandwidths of the receiver, the stronger the noise), so the TDOA measurement error $\sigma_{\text {TDOA }}$ can be deemed to be inversely proportional to the signal bandwidth $B$ and the FDOA measurement error $\sigma_{T D O A}$ is inversely proportional to the duration of accumulated time $T$.
For example, supposing the SNR of input signals $\gamma=0 \mathrm{~dB}$, the noise bandwidth $B_{n}=1 \mathrm{MHz}$, the signal bandwidth (or receiver bandwidth) $B=100 \mathrm{kHz}$, and the accumulated time $T=10 \mathrm{~ms}$, then according to the above expression, the CRLB of the TDOA and FDOA measurement error can be calculated as

$$
\left.\begin{array}{l}
\sigma_{\mathrm{TDOA}}=\frac{0.55}{B} \frac{1}{\sqrt{B_{n} T \gamma}}=\frac{0.55}{100 \times 10^{3}} \frac{1}{\sqrt{10^{6} \times 10^{-2} \times 1}}=55 \mathrm{~ns} \\
\sigma_{\mathrm{FDOA}}=\frac{0.55}{T} \frac{1}{\sqrt{B_{n} T \gamma}}=\frac{0.55}{10 \times 10^{-3}} \frac{1}{\sqrt{10^{6} \times 10^{-2} \times 1}}=0.55 \mathrm{~Hz}
\end{array}\right\}
$$

### 5.5.2.2 Theoretical Accuracy of TDOA-FDOA for Pulse Signal

If the SNR is higher than 10 dB , the estimation variance of the frequency is much closer to the Cramér-Rao bound. Its expression is [10]

$$
\begin{equation*}
\sigma_{f_{d}}=\frac{1}{\pi N_{e} \sqrt{N_{e} \gamma_{e}}} \tag{5.71}
\end{equation*}
$$

where $N_{e}$ is the sampling point of the estimation expression and $\gamma_{e}$ is the equivalent SNR.
Convert the CRLB of the estimated $f_{d}$ into the CRLB of the FDOA measurement error; then [10]

$$
\begin{equation*}
\sigma_{F D O A}=\frac{1}{\pi T_{M} \sqrt{P N \gamma}}, \tag{5.72}
\end{equation*}
$$

Table 5.1 List of the FDOA measurement error with different PRF values

| Precision/PRF | PRF $=100 \mathrm{~Hz}$ | PRF $=1 \mathrm{kHz}$ | PRF $=10 \mathrm{kHz}$ | PRF $=100 \mathrm{kHz}$ |
| :--- | :---: | :---: | :---: | :---: |
| $\sigma_{\text {FDOA }}(\mathrm{Hz})$ | 0.04 | 0.01 | 0.004 | 0.001 |

where $T_{M}$ is the total time of the pulse series (start from the beginning of the first pulse to the end of the last pulse), $T_{M}=P \cdot P R I$ (where PRI is the mean pulse repetition interval), $P$ is the pulse number, and $N$ is the sampling point of a pulse.
For a conventional radar signal, only the time shift is needed. By aligning the two pulses of signals, the number of effective intrapulse sampling points is increased. The time shift has no influence on the accuracy of the FDOA estimation, so its accuracy expression (5.72) can be rewritten as

$$
\begin{equation*}
\sigma_{F D O A}=\frac{1}{\pi T_{M} \sqrt{P N \gamma}} \tag{5.73}
\end{equation*}
$$

These expressions show that with the increase of accumulated time $T_{M}$, the FDOA estimation error decreases and with the increase of intrapulse sampling points $N$, FDOA decreases as the square root of intrapulse sampling points $N$.
For a certain signal, suppose its $B=1 \mathrm{MHz}$, pulse width $=1 \mu \mathrm{~s}$, and $N=60$. For $T=0.1$ second, suppose its PRF $=100$ and its pulse number $P=10$. Therefore, the accuracy of $\sigma_{F D O A}=0.04 \mathrm{~Hz}$ can be calculated. Suppose the sampling frequency is $f_{s}=56 \mathrm{MHz}$; Table 5.1 can then be obtained.
When the pulse repetition frequency (PRF) is low, for example, $\mathrm{PRF}=100 \mathrm{~Hz}$, and if the Doppler shift of signals is larger than the PRF, a problem of FDOA ambiguity may happen. Some other ways may be required to be adopted to solve such problems of FDOA ambiguity.

### 5.5.2.3 Resolution of Time Delay and Frequency Shift

The CAFs of different signals are different and the width of the CAF main peak on the time delay and frequency shift is also different. Here, taking the chirp signal as an example, the use of the width of its CAF main peak as the resolution of the time delay $\triangle$ TDOA and the frequency shift $\triangle$ FDOA gives

$$
\left.\begin{array}{l}
\Delta \mathrm{TDOA}=1 / B  \tag{5.74}\\
\Delta \mathrm{FDOA}=1 / T
\end{array}\right\}
$$

where $T$ is the signal duration time and $B$ is the signal bandwidth at such a time.
When adopting the fast algorithm to calculate piecewise the CAF, the form of the function is almost the same. Thus, resolution of FDOA and TDOA measurements is almost kept the same.

### 5.5.3 Segment Correlation Accumulation Method for CAF Computation

Because the computation load of the CAF is very large, some measures need to be adopted. One of the effective measures is to use the segment correlation accumulation method.

### 5.5.3.1 Segment Correlation Accumulation Method

Supposing the transmitting signal of the emitter is

$$
\begin{equation*}
s(t)=m(t) \mathrm{e}^{\mathrm{j} 2 \pi f_{0} t} \tag{5.75}
\end{equation*}
$$

where $m(t)$ is the complex baseband signal and $f_{0}$ is the signal carrier frequency, then the signal received by the two satellites will be

$$
\begin{align*}
& r_{1}(t)=s\left(\left(1-d_{11}\right) t-d_{10}\right)  \tag{5.76}\\
& r_{2}(t)=s\left(\left(1-d_{21}\right) t-d_{20}\right) \tag{5.77}
\end{align*}
$$

where $d_{11}=v_{1} / c, d_{21}=v_{2} / c, v_{1}$ and $v_{2}$ are the radial velocities of two satellites relative to the emitter, and $c$ is the signal propagation speed,

$$
\begin{align*}
& d_{10}=\left(1-d_{11}\right) t_{1}  \tag{5.78}\\
& d_{20}=\left(1-d_{21}\right) t_{2} \tag{5.79}
\end{align*}
$$

where $t_{1}$ and $t_{2}$ are the propagation times of the signal from the emitter to the satellites.

## Define

$$
\begin{align*}
\tau_{1}(t) & =d_{10}+d_{11} t  \tag{5.80}\\
\tau_{2}(t) & =d_{20}+d_{21} t  \tag{5.81}\\
\tau_{21}(t) & =\tau_{2}(t)-\tau_{1}(t)=d_{20}-d_{10}+\left(d_{21}-d_{11}\right) t=\tau_{0}+\alpha t \tag{5.82}
\end{align*}
$$

where $\tau_{0}=d_{20}-d_{10}$ is the TDOA at $t=0$ of two signals $r_{2}(t)$ and $r_{1}(t), \alpha=d_{21}-d_{11}=$ $\left(\mathrm{v}_{2}-\mathrm{v}_{1}\right) / c \approx v_{0} / f_{0}$, and $v_{0}$ is the Doppler FDOA of two signals.
Substitute expressions (5.76) and (5.77) into expression (5.68) and combine expressions (5.80) and (5.81), which yields

$$
\begin{align*}
C A F(\tau, v) & =\int_{0}^{T} m\left(t-\tau_{1}\right) \mathrm{e}^{j 2 \pi f_{0}\left(t-\tau_{1}\right)} m^{*}\left(t-\tau_{1}-\tau_{21}+\tau\right) \mathrm{e}^{-\mathrm{j} 2 \pi f_{0}\left(t-\tau_{1}-\tau_{21}+\tau\right)} \mathrm{e}^{-j 2 \pi v t} \mathrm{~d} t  \tag{5.83}\\
& =\int_{0}^{T} r_{m}(t, \tau) \mathrm{e}^{j 2 \pi\left(-v t+f_{0} \tau_{21}-f_{0} \tau\right)} \mathrm{d} t=\int_{0}^{T} r_{m}(t, \tau) \mathrm{e}^{\mathrm{j} 2 \pi\left[\left(v_{0}-v\right) t+f_{0}\left(\tau_{0}-\tau\right)\right]} \mathrm{d} t, \tag{5.84}
\end{align*}
$$

where

$$
r_{m}(t, \tau)=m\left(t-\tau_{1}\right) m^{*}\left(t-\tau_{1}-\tau_{21}+\tau\right)
$$

Divide the signal with a duration of $T$ into $K$ segments of equal length, each with a duration of $T / K$, remark as $T_{1}$ and the signal of the $k$ th segment as [11]:

$$
\begin{array}{ll}
r_{1 k}(t)=r_{1}\left(t+k T_{1}\right), & t \in\left[0, T_{1}\right), \\
r_{2 k}(t)=r_{2}\left(t+k T_{1}\right), & t \in\left[0, T_{1}\right) . \tag{5.86}
\end{array}
$$

Define the calculation method of the $\operatorname{CAF}(\tau, v, k)$ of the segment signal as

$$
\begin{equation*}
C A F(\tau, v, k)=\mathrm{e}^{j \phi_{k}} \int_{0}^{T_{1}} r_{1 k}(t) r_{2 k}^{*}(t+\tau) \mathrm{e}^{-j 2 \pi v t} \mathrm{~d} t \tag{5.87}
\end{equation*}
$$

where $\phi_{k}$ is the phase item and its value will be given below.

Substitute expressions (5.85) and (5.86) into expression (5.87) to derivate [11]

$$
\begin{align*}
\operatorname{CAF}(\tau, v, k) & =\mathrm{e}^{j \phi_{k}} \int_{0}^{T_{1}} r_{1}\left(t+k T_{1}\right) r_{2}^{*}\left(t+k T_{1}+\tau\right) \mathrm{e}^{-j 2 \pi v t} \mathrm{~d} t \\
& =\int_{0}^{T_{1}} r_{m}\left(t+k T_{1}, \tau\right) \mathrm{e}^{j 2 \pi\left(-v t+f_{0} \tau_{21 k}-f_{0} \tau+\phi_{k} / 2 \pi\right)} \mathrm{d} t \tag{5.88}
\end{align*}
$$

where

$$
\begin{equation*}
\tau_{21 k}=\tau_{21}\left(t+k T_{1}\right)=\tau_{0}+\alpha\left(t+k T_{1}\right) \tag{5.89}
\end{equation*}
$$

Change expression (5.84) to

$$
\begin{align*}
\operatorname{CAF}(\tau, v) & =\sum_{k=1}^{K} \int_{k T_{1}}^{(k+1) T_{1}} r_{m}(t, \tau) \mathrm{e}^{j 2 \pi\left(-v t+f_{0} \tau_{21}-f_{0} \tau\right)} \mathrm{d} t \\
& =\sum_{k=1}^{K} \int_{0}^{T_{1}} r_{m}\left(t+k T_{1}, \tau\right) \mathrm{e}^{j 2 \pi\left[-v\left(t+k T_{1}\right)+f_{0} \tau_{21 k}-f_{0} \tau\right]} \mathrm{d} t=\sum_{k=1}^{K} \operatorname{CAF}_{k}(\tau, v), \tag{5.90}
\end{align*}
$$

that is,

$$
\begin{equation*}
C A F_{k}(\tau, v)=\int_{0}^{T_{1}} r_{m}\left(t+k T_{1}, \tau\right) \mathrm{e}^{\mathrm{j} 2 \pi\left[-v\left(t+k T_{1}\right)+f_{0} \tau_{21 k}-f_{0} \tau\right]} \mathrm{d} t \tag{5.91}
\end{equation*}
$$

Let $\operatorname{CAF}_{k}(\tau, v)=\operatorname{CAF}(\tau, \nu, k)$ and compare expression (5.88) with expression (5.91); then

$$
\begin{equation*}
\phi_{k}=-2 \pi v k T_{1} . \tag{5.92}
\end{equation*}
$$

When expression (5.92) holds, then

$$
\begin{equation*}
\operatorname{CAF}(\tau, v)=\sum_{k=1}^{K} \operatorname{CAF}(\tau, v, k) \tag{5.93}
\end{equation*}
$$

In the fast algorithm of calculating piecewise CAF, because piecewise CAF is correlation accumulated and equivalent to the overall calculation of the CAF, the FDOA of the algorithm and TDOA measurement accuracy will remain the same.

### 5.5.3.2 Method for Further Decrease of the Computation Load

According to expression (5.87), for the data of the kth segment of the two signals, the discrete expression of the CAF is

$$
\begin{equation*}
\operatorname{CAF}\left(m T_{s}, \frac{p}{Q} F_{s}, k\right)=\exp \left(j 2 \pi \frac{k p N}{Q}\right) \sum_{n=k N}^{k N+(N-1)} r_{1}(n) r_{2}^{*}(n+m) \exp \left(-j 2 \pi \frac{p n}{Q}\right) \tag{5.94}
\end{equation*}
$$

where $N$ is the number of sampling points in the kth segment data, $T_{s}$ is the sampling period, the time delay $\tau=m T_{s}, F_{s}=1 / T_{s}$ is the sampling frequency, and $F_{s} / Q$ is the resolution unit of the frequency shift.

The FFT (fast Fourier transform) calculation is adopted by expression (5.94). Let $Q=N$; then its computation load (number of complex multiplications) is about

$$
\begin{equation*}
G_{1} \approx M\left(N \log _{2} N+2 N\right) \tag{5.95}
\end{equation*}
$$

where $M$ is the number of resolution units of time delay.
In order to further decrease the computation load, the following approximation is made to expression (5.94):

$$
\begin{equation*}
\operatorname{CAF}\left(m T_{s}, \frac{p}{N_{1}} F_{s}, k\right) \approx \exp \left(j 2 \pi k p N_{2}\right) \sum_{l=0}^{N_{2}-1} u_{1}(l, m) \exp \left(-j 2 \pi \frac{p l}{N_{2}}\right), \tag{5.96}
\end{equation*}
$$

where

$$
\begin{equation*}
u_{1}(l, m)=\frac{1}{N_{1}} \sum_{n=l N_{1}}^{l N_{1}+N_{1}-1} r_{1}(n) r_{2}^{*}(n+m) \quad\left(l=0,1, \ldots, N_{2}-1\right), \tag{5.97}
\end{equation*}
$$

that is, divide the $N$-point instantaneous correlation function $r_{1}(n) r_{2}^{*}(n+m)$ into $N_{2}$ segments, calculate the average value of $N_{1}$ points for each segment, and then calculate the CAF of $N_{1}$ points. Now the computation load is decreased to

$$
\begin{equation*}
G_{2} \approx M\left(N_{2} \log _{2} N_{2}+N_{2}+N\right), \tag{5.98}
\end{equation*}
$$

that is, the computation load is decreased by $N_{1}$ times.
Calculate the CAF according to expression (5.93) in one time segment and suppose the total number of samples is $N_{a}=K N=K N_{1} N_{2}$; then the total number of complex multiplications of CAF calculation is about

$$
\begin{equation*}
G_{3.19} \approx K G_{2} \approx M\left(K N_{2} \log _{2} N_{2}+K N_{2}+K N\right) \approx M N_{a} \frac{1}{N_{1}}\left(\log _{2} \frac{N_{a}}{K N_{1}}+1+N_{1}\right) \tag{5.99}
\end{equation*}
$$

Calculate the CAF according to expression (5.68) in one time segment, suppose the total number of samples is $N_{a}$, and use the FFT calculation method; then the total number of complex multiplications is about

$$
\begin{equation*}
G_{3.1} \approx M N_{a} \log _{2} N_{a} . \tag{5.100}
\end{equation*}
$$

The number of complex multiplications using the calculation method of expression (5.93) is the same as that of expression (5.99). The computation load ratio $d_{\text {comp }}$ between the two calculation methods is

$$
\begin{equation*}
d_{\text {comp }} \approx \frac{N_{1} \log _{2} N_{a}}{N_{1}+\log _{2} N_{a}-\log _{2} N_{1}-\log _{2} K} \tag{5.101}
\end{equation*}
$$

Certainly, in the above analysis, the time-consuming additive operation is ignored. For the floating-point DSP (digital signal processor) or field-programmable gate array (FPGA), the time consumption of floating-point addition and floating-point multiplication is almost the same.


Figure 5.21 Graph of the single-signal CAF

### 5.5.4 Resolution of Multiple Signals of the Same Time and Same Frequency

In Section 5.5.1, the method of CAF is used for TDOA and FDOA estimations. Next, we will discuss the multisignal resolution performance of this method in detail.
As shown in Figure 5.21, for a single signal, there are three kinds of lobes: one is the signal main lobe, whose corresponding time and frequency value is the estimated value of the true TDOA and FDOA; the second is the signal side lobe, of which there are generally several; and the last one is the noise lobe. The signal side lobe is generally lower than the signal main lobe. When the SNR is low, the noise lobe may exceed the signal side lobe or even the signal main lobe. In order to detect the signal main lobe from intensity and ensure the pre-set TDOA and FDOA estimation accuracy, the SNR must be higher than a certain threshold. This is a necessary condition for single-signal detection as well as the precondition of multisignal resolution.
In addition, the relative intensity of signals is also an important factor influencing signal resolution. We will discuss signal resolution for the following two cases by taking two aliasing signals A and B as an example.
Suppose the power of signal A is $P_{A}$ and the power of signal B is $P_{B}$ without loss of generality and suppose $P_{A}>P_{B}$.

Case 1. $P_{A}>P_{B}$ but their difference is not great, and the main lobe of signal B is weaker than that of signal A , but much stronger than the side lobe and background noise lobe of signal A .

In this case, even if the main lobe of signal B is aliasing with the side lobe or background noise lobe of signal A, the mixed lobe will not deviate from the position of the main lobe of signal B, so the mixed lobe can be equivalent to the main lobe of signal B . That is to say, the resolution of the main lobe of signal B and the side lobe and background noise lobe of signal A is not required to be considered; only the resolution of the main lobe of signal B and the main lobe of signal A is required.

When the distance between the main lobes of the two signals is larger than the resolution distance, the lobes of the two signals can be resolved on the CAF graph, to obtain the correct estimated TDOA-FDOA value of each signal. On the contrary, if the distance between the two main lobes is shorter than the resolution distance, the two lobes will fuse together and form a mixed lobe with an estimation of the TDOA-FDOA value between the two lobes, which deviates the stronger one, as shown in Figure 5.22.
Case 2. $P_{A}>P_{B}$ and their difference is large. The main lobe of signal B is weaker than that of signal A , but its intensity is close to that of the side lobe of signal A .

In this case, two things need to be considered. One is the resolution of the main lobes of signal B and signal A, and the other is the influence of the side lobe of signal A on the main lobe of signal B .

When the distance between the main lobes of signal A and signal B is larger than the resolution distance, the two lobes can be resolved on the CAF graph, to obtain the correct estimated TDOA-FDOA value of each signal. On the contrary, if the distance between the two lobes is shorter than the resolution distance, the two lobes will fuse to form a mixed lobe with location on the TDOA-FDOA surface between the two lobes, deviating the stronger one, as shown in Figure 5.23.

The influence of the side lobe of signal A on the main lobe of signal B has the following effects. If the distance between the two lobes is larger than the resolution distance, the influence of the side lobe of signal A on the main lobe of signal B is small; if the distance between the two lobes is smaller than the resolution distance, two lobes will be combined to a mixed lobe and the TDOA and FDOA estimation will be somewhere between the two lobes. The smaller the distance between the two lobes, the smaller is the strength ratio of the signal B main lobe to the signal A side secondary lobe and the larger is the deviation of the mixed lobe relative to the signal B main lobe. If the location of the mixed lobe is used to represent that of the signal B main lobe, a bias will be caused.


Figure 5.22 Graph of the CAF when the two signals have almost the same power


Figure 5.23 Graph of the CAF when the power of the two signals is different

We can obtain the following conclusions:

1. The CAF method has a certain multisignal resolution capability. Among the TDOA distance and FDOA distance of two signals, if either one is larger than the corresponding resolution, the two signals can be resolved in the graph of CAF.
2. Generally speaking, the TDOA resolution between the two signals is determined by their bandwidth and is not easy to be changed. The FDOA resolution between the two signals is determined by the integration time of the signal, which is controllable. The increase in cumulative time of the signal will help to improve the multisignal resolution performance.

## References

1. Pattison, T. and Chou, S.I. (2000) Sensitivity analysis of dual-satellite geolocation. IEEE Transactions on Aerospace and Electronic Systems, 36 (1), 56-71.
2. Guo, F. (2008) Analysis of dual-satellites TDOA-FDOA passive localization. Aerospace Electronic Warfare, 22: 20-23 (in Chinese).
3. Guo, F. and Fan, Y. (2008) Combined dual-satellite TDOA and FDOA localization and its error analysis. Journal of Astronautics, 29(4): 1381-1386 (in Chinese).
4. Huang, Z. and Lu, J. (2003) Space-based passive localization and modern small satellite technology. Journal of the Academy of Equipment Command and Technology, 14, 24-29.
5. Ho, K.C.a. and Chan, Y.T. (1997) Geolocation of a known altitude object from TDOA and FDOA measurements. IEEE Transactions on Aerospace and Electronic Systems, 33 (3), 770-783.
6. Sun, Z. (2006) Localization Technology Research on Upstream Signals of Geostationary Satellites. Zhengzhou: University of Information Engineering (in Chinese).
7. Qu, W., Ye S., and Sun, Z. (2005) Algorithm of location iteration for satellite interference location. Journal of Electronics and Information Technology, 5(27): 797-800 (in Chinese).
8. Foy, W.H. (1976) Position-location solutions by Taylor-series estimation. IEEE Transactions on Aerospace and Electronic Systems, AES-12 (2), 187-194.
9. Stein, S. (1981) Algorithms for ambiguity function processing. IEEE Transactions on Acoustics Speech and Signal Processing, ASSP-29 (3), 588-598.
10. Zhang, Y. (2008) Research on Key Technologies of Dual LEO Satellite TDOA-FDOA Passive Localization System. Changsha: Graduate School of National University of Defense Technology (in Chinese).
11. Ulman, R. and Geraniotis, E. (1999) Wideband TDOA-FDOA processing using summation of short-time CAFs. IEEE Transactions on Signal Processing, 12 (47), 3193-3200.

## 6

## Single-Satellite Geolocation System Based on the Kinematic Principle

As a satellite orbits the earth at a high speed, parameters such as signal frequency, time of arrival (TOA), and phase rate measured by the interferometer always change with time. However, such variation is correlated with emitter locations on the earth's surface. For example, as relative motion always exists between the satellite and the target emitter, the signal received by the satellite is affected by the Doppler effect. Obviously, the signal instantaneous carrier frequency or the pulse repetition interval (PRI) for the pulse signal changes with time, and the change of signal frequency produces parameters such as the frequency change rate. These parameters essentially reflect relations of relative location and motion between the satellite and the target emitter at that time. By utilizing the satellite-borne navigation system or the ground TTC\&M (telemetry, tracking, command, and monitoring) system, the position and speed of the satellite can be obtained. Considering the fixed target emitter on the earth's surface, the expressions of parameters mentioned above are actually expressions in relation to unknown parameters like the emitter position.

### 6.1 Single-Satellite Geolocation Model

Assume that the coordinates of an ground emitter in the ECEF (earth-center earth-fixed) coordinates system are $(x, y, z)$ and that a satellite can intercept the signal $N$ times from $t_{1}$ to $t_{N}$. Suppose the position $\left(x_{i}, y_{i}, z_{i}\right)$, velocity $\mathbf{v}_{\mathbf{i}}$, and acceleration $\mathbf{a}_{\mathbf{i}}$ of the satellite at the $i$ th moment can be measured by the satellite navigation system or the ground TTC\&M system, but the position $(x, y, z)$, signal carrier frequency $f_{0}$, or pulse PRI of the signal target emitter transmitted will be unknown. Figure 6.1 shows the relations between the known and unknown elements.


Figure 6.1 Sketch of localization based on the kinematic principle

Suppose that $\theta$ is a parameter measured by a single-satellite geolocation system obtained at the moment $i$ :

$$
\begin{equation*}
\widehat{\theta}_{i}=f\left(x, y, z, \theta_{0}\right)+\xi_{i} \quad(i=1,2, \ldots, N) \tag{6.1}
\end{equation*}
$$

where $\theta_{0}$ represents the real value of the emitter transmitted signal and $\xi_{i}$ is the measurement error. If $\widehat{\theta}_{i}$ can be measured with different times $i=1,2, \ldots, N$ and at various positions on the satellite orbit, then we can acquire a system of expressions. Theoretically, where the number of expressions in the system is larger than that of the unknown numbers, the position coordinates of the emitter can be estimated. As a result, the geolocation of the emitter is realized.
As the emitter on the earth's surface or in the air moves much slower than the satellite, so the emitter can be deemed as static during a small period when the LEO (low earth orbit) satellite is passing by. We therefore assume that the emitter is fixed on the earth's surface in the following analysis of the article. The coordinates of the emitter on the earth's surface are also supposed to match the earth surface expression, for example, that of the WGS-84 ellipse surface model is

$$
\begin{equation*}
g(x, y, z)=0 \tag{6.2}
\end{equation*}
$$

If the earth's surface in Equation (6.2) or (2.30) is taken into account in the geolocation, theoretically, of the minimum number of observations the demand can be reduced by one and geolocation accuracy is improved at the same time. Each kind of single-satellite geolocation system based on the kinematics explored in this chapter all take the earth's surface as a prior known localizing surface.

### 6.2 Single-Satellite Single-Antenna Frequency-Only Based Geolocation

### 6.2.1 Frequency-Only Based Geolocation Method

According to the Doppler effect resulting from relative motion between the satellite and the emitter, the instantaneous frequency of the intercepted signal is

$$
\begin{equation*}
f=f_{0}+f_{d}, \tag{6.3}
\end{equation*}
$$

where the Doppler frequency is

$$
\begin{equation*}
f_{d}=-f_{0} \frac{\dot{r}(t)}{c} \tag{6.4}
\end{equation*}
$$

Here $\dot{r}(t)$ is the derivative of the range between the satellite and the emitter, or the radial velocity of relative motion. According to various definitions in Section 6.1, assume that $\mathbf{u}_{i}=\mathbf{r}_{i} /\left\|\mathbf{r}_{i}\right\|$ is the unit vector of the line of sight (LOS). Then $\dot{r}_{i}(t)$ at the moment $i$ can be related to the emitter coordinates by the following expression:

$$
\begin{equation*}
\dot{r}_{i}(t)=\mathbf{u}_{i}^{\mathrm{T}} \cdot \mathbf{v}_{i}=\frac{v_{x i}\left(x_{i}-x\right)+v_{y i}\left(y_{i}-y\right)+v_{z i}\left(z_{i}-z\right)}{r_{i}} \tag{6.5}
\end{equation*}
$$

Where

$$
r_{i}=\sqrt{\left(x_{i}-x\right)^{2}+\left(y_{i}-y\right)^{2}+\left(z_{i}-z\right)^{2}}
$$

According to expressions (6.3) and (6.5), therefore, the signal frequency received by the satellite at the moment $i$ is

$$
\begin{equation*}
\widehat{f}_{i}=f_{0}\left[1-\frac{v_{x i}\left(x_{i}-x\right)+v_{y i}\left(y_{i}-y\right)+v_{z i}\left(z_{i}-z\right)}{c r_{i}}\right]+\xi_{i}=f_{0} f_{i}(x, y, z)+\xi_{i}, \tag{6.6}
\end{equation*}
$$

where, as defined above, $\xi_{i}$ is the noise for the frequency measurement at the moment $i$, and the noise for the frequency measurement at each moment is subject to independent Gaussian distribution. Frequencies measured at the total of $N$ moments are expressed in matrix form as

$$
\begin{equation*}
\boldsymbol{\Psi}=f_{0} \cdot \mathbf{F}+\boldsymbol{\xi} \tag{6.7}
\end{equation*}
$$

where $\boldsymbol{\Psi}=\left[\hat{f}_{1}, \ldots, \widehat{f}_{i}, \ldots, \widehat{f}_{N}\right]^{\mathrm{T}}, \quad \mathbf{F}=\left[f_{1}(x, y, z), \ldots, f_{i}(x, y, z), \ldots, f_{N}(x, y, z)\right]^{\mathrm{T}}$, and $\boldsymbol{\xi}=\left[\xi_{1}, \ldots, \xi_{i}, \ldots, \xi_{N}\right]^{\mathrm{T}}$.
Replace the position coordinates ( $x, y, z$ ) of the emitter in $\mathbf{F}$ with transformation of expression (2.30); then we get the matrix expression in relation to the geodetic coordinates $(L, B)$ of the emitter ( $H$ is assumed to be known by prior information). Only four variables are unknown. Thus, the static emitter on the earth's surface can be localized [1], theoretically, if $N \geq 4$. Now carry out a grid search in the area containing the emitter by longitude and latitude. Define $\sum$ as a 2D grid set of points. For each grid point $\left(L_{k}, B_{k}\right) \in \sum$, count the least squares solution of $f_{0}$ :

$$
\begin{equation*}
\widehat{f}_{k}=\left(\mathbf{F}_{k}^{\mathrm{T}} \mathbf{F}_{k}\right)^{-1} \mathbf{F}_{k}^{\mathrm{T}} \boldsymbol{\Psi}, \tag{6.8}
\end{equation*}
$$

where $\mathbf{F}_{k}=\left[f_{1}\left(L_{k}, B_{k}\right), \ldots, f_{i}\left(L_{k}, B_{k}\right), \ldots, f_{N}\left(L_{k}, B_{k}\right)\right]^{\mathrm{T}}$.
Substitute the result of expression (6.8) into the expression below, and solve the expression by the grid search method:

$$
\begin{equation*}
J\left(L_{k}, B_{k}\right)=\left\|\boldsymbol{\Psi}-\widehat{f}_{k} \cdot \mathbf{F}_{k}\right\|^{2} \tag{6.9}
\end{equation*}
$$

Construct a cost function. Then the least minimum of $J\left(L_{k}, B_{k}\right)$ is the optimum estimate of the emitter location.

### 6.2.2 Analysis of the Geolocation Error

As errors always exist in frequency measurement, we are now going to derive the Cramér-Rao lower bound (CRLB) in the case when the position and frequency of the emitter are being estimated and the transmitted frequency of the emitter is unknown.
Assume that $N$ independent frequency measurements have been performed and the frequency measurement error complies with zero-mean, independent Gaussian distributions, that is, $\xi_{1}, \ldots, \xi_{i}, \ldots, \xi_{N} \rightarrow \mathcal{N}\left(0, \sigma^{2}\right)$. Then the joint probability density of $f_{1}, \ldots, f_{N}$ is

$$
\begin{equation*}
p\left(f_{1}, \ldots, f_{N} ; L, B, f_{0}\right)=\frac{1}{\left(2 \pi \sigma^{2}\right)^{N / 2}} \exp \left\{-\frac{\sum_{i=1}^{N}\left[f_{i}-f_{i}\left(L, B, f_{0}\right)\right]^{2}}{2 \sigma^{2}}\right\} \tag{6.10}
\end{equation*}
$$

where $f_{i}\left(L, B, f_{0}\right)=f_{0} f_{i}(L, B)$. Take the logarithm of both sides of expression (6.10). The result is

$$
\begin{equation*}
\ln (p)=-\frac{\sum_{i=1}^{N}\left[f_{i}-f_{i}\left(L, B, f_{0}\right)\right]^{2}}{2 \sigma^{2}}-\frac{N}{2} \ln \left(2 \pi \sigma^{2}\right) . \tag{6.11}
\end{equation*}
$$

Let $\boldsymbol{\theta}=\left[L, B, f_{0}\right]^{\mathrm{T}}$; then the derivative of the above expression about $\boldsymbol{\theta}$ is

$$
\frac{\partial \ln (p)}{\partial \boldsymbol{\theta}}=\frac{1}{\sigma^{2}}\left[\begin{array}{l}
\sum_{i=1}^{N}\left[f_{i}-f_{i}\left(L, B, f_{0}\right)\right] \frac{\partial f_{i}\left(L, B, f_{0}\right)}{\partial L}  \tag{6.12}\\
\sum_{i=1}^{N}\left[f_{i}-f_{i}\left(L, B, f_{0}\right)\right] \frac{\partial f_{i}\left(L, B, f_{0}\right)}{\partial B} \\
\sum_{i=1}^{N}\left[f_{i}-f_{i}\left(L, B, f_{0}\right)\right] \frac{\partial f_{i}\left(L, B, f_{0}\right)}{\partial f_{0}}
\end{array}\right]
$$

We therefore get the Fisher information matrix

$$
\begin{equation*}
[\mathbf{I}(\boldsymbol{\theta})]_{m n}=-E\left[\frac{\partial^{2} \ln (p)}{\partial \theta_{m} \partial \theta_{n}}\right]=\frac{1}{\sigma^{2}} \mathbf{H}^{\mathrm{T}} \mathbf{H} \tag{6.13}
\end{equation*}
$$

where

$$
\mathbf{H}=\left[\begin{array}{lll}
\frac{\partial f_{1}\left(L, B, f_{0}\right)}{\partial L} & \cdots & \frac{\partial f_{N}\left(L, B, f_{0}\right)}{\partial L}  \tag{6.14}\\
\frac{\partial f_{1}\left(L, B, f_{0}\right)}{\partial B} & \cdots & \frac{\partial f_{N}\left(L, B, f_{0}\right)}{\partial B} \\
\frac{\partial f_{1}\left(L, B, f_{0}\right)}{\partial f_{0}} & \cdots & \frac{\partial f_{N}\left(L, B, f_{0}\right)}{\partial f_{0}}
\end{array}\right]^{\mathrm{T}} .
$$

According to the transformational relation of expression (2.30), we know that $f_{0}$ does not need to be transformed, so the parameter transformation matrix can be built as follows:

$$
\left[\begin{array}{c}
x  \tag{6.15}\\
y \\
z \\
f_{0}
\end{array}\right]=\boldsymbol{\phi}(\boldsymbol{\theta})=\left[\begin{array}{c}
N_{t} \cos B \cos L \\
N_{t} \cos B \sin L \\
N_{t}\left(1-e^{2}\right) \sin B \\
f_{0}
\end{array}\right] .
$$

According to the CRLB definition of vector parameter transformation [2], the CRLB is

$$
\begin{equation*}
C R L B_{\left(x, y, z, f_{0}\right)}=\frac{\partial \boldsymbol{\phi}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \mathbf{I}^{-1}(\boldsymbol{\theta}) \frac{\partial \boldsymbol{\phi}(\boldsymbol{\theta})^{\mathrm{T}}}{\partial \boldsymbol{\theta}}=\sigma^{2} \operatorname{tr}\left(\frac{\partial \boldsymbol{\phi}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\left(\mathbf{H}^{\mathrm{T}} \mathbf{H}\right)^{-1} \frac{\partial \boldsymbol{\phi}(\boldsymbol{\theta})^{\mathrm{T}}}{\partial \boldsymbol{\theta}}\right) \tag{6.16}
\end{equation*}
$$

where $\operatorname{tr}(\cdot)$ represents the trace of the matrix.

### 6.2.3 Analysis of the Frequency-Only Based Geolocation Error

Assume that the carrier frequency of the emitter signal $f_{0}=1.3 \mathrm{GHz}$ and $f_{0}=130 \mathrm{MHz}$. The position, speed, and acceleration of the satellite with total time of $T=60$ seconds is obtained from the ephemeris report by STK ${ }^{\circledR}$ software (satellite tool kit). Then compute the lower bounds of the error of frequency-only based geolocation methods under this condition. Take the RMS (root mean square) of the frequency measurement error as the theoretical lower bound value $\sigma_{f_{0}} \approx 0.053 \mathrm{~Hz}$; then we get a GDOP (geometric dilution of precision) graph as shown in Figure 6.2a under the above condition, where the unit of geolocation error in the graph is the kilometer. The bold line in the middle of the graph is the subsatellite track during 1 minute. Obviously, the equal-error curves are basically in the form of a symmetric distribution on both sides of the subsatellite track. The best effect of geolocation appears on both sides of the subsatellite track, while the worst occurs near the subsatellite point.
From Figure 6.2a we know that when accuracy of the frequency measurement is high, this geolocation method enjoys a small error in a considerably large area. In practice, the error of frequency measurement will be larger and the geolocation error will increase due to factors like the geolocation method. For an area monitoring reconnaissance satellite, however, accuracy to this extent is still acceptable in some cases.
We know that the signal frequency imposes an influence on geolocation accuracy from the analysis in Section 6.2.2. Therefore, we now assume that the signal frequency is lowered to be $1 / 10$ of the original but the variance of frequency measurement error remains the same; this gives a GDOP graph as shown in Figure 6.2b.


Figure 6.2 GDOP graph for the single-satellite frequency-only based geolocation: (a) $f_{0}=1.3 \mathrm{GHz}$ and (b) $f_{0}=130 \mathrm{MHz}$

Table 6.1 Theoretical lower bound of the geolocation error at a certain point under conditions of different accuracy of frequency measurements

| RMS of frequency <br> measurement error (Hz) | 0.053 | 0.53 | 5.3 | 53 | 530 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Geolocation error (km) | 0.004285 | 0.04285 | 0.4285 | 4.285 | 42.85 |

Obviously, when the frequency is lowered to be $1 / 10$ of the original, the geolocation error will be 10 times larger. From this, it is known that when accuracy of the frequency measurement of the receiver can remain the same, a better geolocation performance occurs for a signal of high frequency.
Choose a certain point with coordinates of $\left(25^{\circ} \mathrm{N}, 137.5^{\circ} \mathrm{E}\right)$ and a signal frequency $f_{0}=1.3 \mathrm{GHz}$. Now analyze the lower bounds of the error when the supposed emitter at this point is being localized under a scenario of different amounts of accuracy of frequency measurement. As shown in Table 6.1, in the same scene as aforesaid, choose a series of frequency measurement RMS errors, where the theoretical lower bound values of the geolocation error at this point are obtained separately.
From Table 6.1 and expression (6.16), the increase of geolocation error is shown to be in proportion to the frequency measurement error. Therefore, improvement in frequency measurement accuracy is an important approach to use to decrease the geolocation error.
According to the process of calculation, when the satellite is on the zenith of a subsatellite track center, the distance between it and the emitter is about 645.17 km . Therefore, when the signal $f_{0}$ is 1.3 GHz and accuracy of the frequency measurement is under about 70 Hz , the geolocation error will be less than approximately 6.5 km . This is acceptable.

### 6.3 Single-Satellite Geolocation by the Frequency Changing Rate Only

### 6.3.1 Model of Geolocation by the Frequency Changing Rate Only

In the last section of frequency-only geolocation, an accurate value of frequency is not, in fact, necessary before the process of geolocation; however, it must be evaluated during the process of geolocation. What is really useful in the process of frequency measurement geolocation is the changing information of the frequency. Therefore, if the frequency changing rate (FCR) in the whole process is measured to execute geolocation, then estimation of the signal frequency in an exact way is unnecessary (only a rough estimation of frequency is needed to obtain the wavelength $\lambda$ measurement). According to the principle of kinematics of particles [3], essentially the changing rate of signal frequency reflects the radial acceleration of the relative motion between the satellite and the target emitter:

$$
\begin{equation*}
a=-\frac{\ddot{r}_{i}}{\lambda} \tag{6.17}
\end{equation*}
$$

where $\ddot{r}_{i}$ is the radial acceleration of relative motion at the moment $i$. According to kinematics principles, the radial velocity is [3]

$$
\begin{equation*}
\dot{r}=\mathbf{u}^{\mathrm{T}} \cdot \mathbf{v}=\frac{\mathbf{r}^{\mathrm{T}} \cdot \mathbf{v}}{r} \tag{6.18}
\end{equation*}
$$

Taking the derivative of this, we get

$$
\begin{align*}
\ddot{r}=\frac{\mathrm{d} \dot{r}}{\mathrm{~d} t} & =\frac{r\left(\frac{\mathrm{~d} \mathbf{r}^{\mathrm{T}}}{\mathrm{~d} t} \cdot \mathbf{v}+\mathbf{r}^{\mathrm{T}} \cdot \frac{\mathrm{~d} \mathbf{v}}{\mathrm{~d} t}\right)-\left(\mathbf{r}^{\mathrm{T}} \cdot \mathbf{v}\right) \frac{\mathrm{d} r}{\mathrm{~d} t}}{r^{2}} \\
& =\frac{r\left(\mathbf{v}^{\mathrm{T}} \cdot \mathbf{v}+\mathbf{r}^{\mathrm{T}} \cdot \mathbf{a}\right)-\left(\mathbf{r}^{\mathrm{T}} \cdot \mathbf{v}\right) \frac{\mathbf{r}^{\mathrm{T}} \cdot \mathbf{v}}{r}}{r^{2}}, \\
& =\frac{\left(\mathbf{v}^{\mathrm{T}} \cdot \mathbf{v}+\mathbf{r}^{\mathrm{T}} \cdot \mathbf{a}\right)}{r}-\frac{\left(\mathbf{r}^{\mathrm{T}} \cdot \mathbf{v}\right)^{2}}{r^{3}} \tag{6.19}
\end{align*}
$$

where $\mathbf{v}$ and $\mathbf{a}$ are the velocity vector and acceleration vector of the satellite separately, and in practical applications of single-satellite geolocation, they and the position of the satellite are known; $\mathbf{r}=\left[x_{s}-x, y_{s}-y, z_{s}-z\right]^{\mathrm{T}}$ is the relative motion vector of the two. Note that ( $x_{s}, y_{s}, z_{s}$ ) are the position components of the satellite in the ECEF coordinates system and $(x, y, z)$ are the position coordinates of the target emitter in the ECEF coordinates system.
It is known from expressions (6.17) and (6.19) that the FCR acquired by measuring parameters of the received signal at any $i$ th moment is

$$
\begin{equation*}
\widehat{\alpha}_{i}=\frac{1}{\lambda}\left[\frac{\left(\mathbf{r}_{i}^{\mathrm{T}} \cdot \mathbf{v}_{i}\right)^{2}}{r_{i}^{3}}-\frac{\mathbf{v}_{i}^{\mathrm{T}} \cdot \mathbf{v}_{i}+\mathbf{r}_{i}^{\mathrm{T}} \cdot \mathbf{a}_{i}}{r_{i}}\right]+\xi_{i} \quad(i=1, \ldots, N) \tag{6.20}
\end{equation*}
$$

The expression reflects the relation between observations and the unknown value (emitter position). In the expression, $\xi_{i}$ means the error of the measured FCR , and $\lambda$ is the wavelength of the signal, which can be obtained by the coarse frequency measurement $f_{0}$. Though it may cause error, taking the relational expression $\lambda=c / f_{0}$ of the wavelength and frequency gives

$$
\begin{equation*}
\mathrm{d} \lambda=-\frac{c \mathrm{~d} f_{0}}{f_{0}^{2}} \tag{6.21}
\end{equation*}
$$

It is clear that the error of frequency measurement has little influence on the wavelength.
Suppose the position $\left(x_{i}, y_{i}, z_{i}\right)$, velocity ( $v_{x i}, v_{y i}, v_{z i}$ ), and acceleration ( $a_{x i}, a_{y i}, a_{z i}$ ) of the satellite at the moment $i$ are known. Assume that the longitude and latitude of the emitter is $\left(B_{t}, L_{t}\right)$; then $\mathbf{r}_{i}$ can be expressed according to expression (2.30) as

$$
\begin{equation*}
\mathbf{r}_{i}=\left(x_{i}-N_{t} \cos B_{t} \cos L_{t}, y_{i}-N_{t} \cos B_{t} \sin L_{t}, z_{i}-N_{t}\left(1-e^{2}\right) \sin B_{t}\right) . \tag{6.22}
\end{equation*}
$$

Therefore, expression (6.20) can be written as

$$
\begin{equation*}
\widehat{\alpha}_{i}=\frac{1}{\lambda} g_{i}(B, L)+\xi_{i} \quad(i=1, \ldots, N), \tag{6.23}
\end{equation*}
$$

where $g_{i}(B, L)$ is a function of the radial acceleration with a format like Equation (6.20). Turning expression (6.23) into the format of a matrix, we get

$$
\begin{equation*}
\boldsymbol{\alpha}=\frac{1}{\lambda} \mathbf{G}(B, L)+\boldsymbol{\xi}, \tag{6.24}
\end{equation*}
$$

where $\boldsymbol{\alpha}=\left[\hat{\alpha}_{1}, \ldots, \hat{\alpha}_{i}, \ldots, \hat{\alpha}_{N}\right]^{\mathrm{T}}, \mathbf{G}(B, L)=\left[g_{1}(L, B), \ldots, g_{i}(L, B), \ldots, g_{N}(L, B)\right]^{\mathrm{T}}$, and $\boldsymbol{\xi}=\left[\xi_{1}, \ldots, \xi_{i}, \ldots, \xi_{N}\right]^{\mathrm{T}}$.
Perform grid searching in the same way as described in Section 6.2.1. For each grid $\left(L_{k}, B_{k}\right) \in \sum$, calculate the cost function

$$
\begin{equation*}
J\left(L_{k}, B_{k}\right)=1 /\left\|\boldsymbol{\alpha}-\frac{1}{\lambda} \mathbf{G}_{k}\right\|^{2} . \tag{6.25}
\end{equation*}
$$

In the expression $\mathbf{G}_{k}=\left[g_{1}\left(L_{k}, B_{k}\right), \ldots, g_{i}\left(L_{k}, B_{k}\right), \ldots, g_{N}\left(L_{k}, B_{k}\right)\right]^{\mathrm{T}}$. Choose the point that maximizes the cost function as an optimum estimate of the emitter position.
According to expression (6.25), calculate the cost function at each grid point and obtain cost functions of all grid points. A 3D distribution graph of the cost functions example is shown as Figure 6.3.
After the cost function $J\left(L_{k}, B_{k}\right)$ is obtained as Figure 6.3, separately search the points with the maximum value of cost function at the left lower and right lower parts of the figure and find the coordinates of the corresponding maximum grid points. Change the coordinates into those of longitude and latitude and record them separately as $\left(\hat{x}_{u}, \hat{y}_{u}\right)$ and $\left(\hat{x}_{d}, \hat{y}_{d}\right)$. There are two possible geolocation points. Normally, the subsatellite track of the observer is the axis of the two symmetrical points, and one point is on the left side of the subsatellite track (left geolocation point) and the other is on the right side (right geolocation point).
Because of ambiguity of the two geolocation points caused by the method, other information must be used to eliminate the geolocation ambiguity. Similar methods, such as given in Section 5.2.4, are suggested to increase the direction finding information that can be used to solve ambiguity.


Figure 6.3 Graph of the cost function $J\left(L_{k}, B_{k}\right)$

### 6.3.2 CRLB of the Geolocation Error

As measurement accuracy of the wavelength is comparatively high, the influence of the estimation error of wavelength $\lambda$ can be neglected. Assuming that each error of the measured FCR is mutually independent and complies with $\xi_{1}, \ldots, \xi_{i}, \ldots, \xi_{N} \rightarrow \mathcal{N}\left(0, \sigma^{2}\right)$, the joint probability density function of $\alpha_{1}, \ldots, \alpha_{N}$ is

$$
\begin{equation*}
p\left(\alpha_{1}, \ldots, \alpha_{N} ; L, B\right)=\frac{1}{\left(2 \pi \sigma^{2}\right)^{N / 2}} \exp \left\{-\frac{\sum_{i=1}^{N}\left[\alpha_{i}-\frac{1}{\lambda} g_{i}(L, B)\right]^{2}}{2 \sigma^{2}}\right\} \tag{6.26}
\end{equation*}
$$

Then the Fisher information matrix is

$$
\begin{equation*}
\mathbf{I}(\boldsymbol{\theta})=\frac{1}{\lambda^{2} \sigma^{2}} \mathbf{H}^{\mathrm{T}} \mathbf{H} \tag{6.27}
\end{equation*}
$$

where $\boldsymbol{\theta}=[B, L]^{\mathrm{T}}$ and

$$
\mathbf{H}=\left[\begin{array}{lll}
\frac{\partial g_{1}(L, B)}{\partial L} & \cdots & \frac{\partial g_{N}(L, B)}{\partial L}  \tag{6.28}\\
\frac{\partial g_{1}(L, B)}{\partial B} & \cdots & \frac{\partial g_{N}(L, B)}{\partial B}
\end{array}\right]^{\mathrm{T}}
$$



Figure 6.4 GDOP graph for the single-satellite frequency changing rate only geolocation: (a) $f_{0}=1.3 \mathrm{GHz}$ and (b) $f_{0}=130 \mathrm{MHz}$

After variable transformation, the CRLB in each direction of $(x, y, z)$ is

$$
\begin{equation*}
C R L B_{(x, y, z)}=\frac{\partial \boldsymbol{\phi}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \mathbf{I}^{-1}(\boldsymbol{\theta}) \frac{\partial \boldsymbol{\phi}(\boldsymbol{\theta})^{\mathrm{T}}}{\partial \boldsymbol{\theta}}=\sigma^{2} \lambda^{2} \operatorname{tr}\left(\frac{\partial \boldsymbol{\phi}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\left(\mathbf{H}^{\mathrm{T}} \mathbf{H}\right)^{-1} \frac{\partial \boldsymbol{\phi}(\boldsymbol{\theta})^{\mathrm{T}}}{\partial \boldsymbol{\theta}}\right), \tag{6.29}
\end{equation*}
$$

where $\operatorname{tr}(\cdot)$ represents the trace of the matrix.

### 6.3.3 Geolocation Simulation

Use the same geolocation scenario as described in Section 6.2.3, where the error RMS of the FCR measured is $\sigma_{\alpha} \approx 1 \mathrm{~Hz} / \mathrm{s}$. The GDOP graph of the same subsatellite area with different frequencies is shown in Figure 6.4.
It is clear that the error of the measured FCR is higher than that of the frequency measurement and the geolocation performance of the former is poorer than the latter.
From Figure 6.4 it can be seen that for the same accuracy of FCR measured, the geolocation error will be increased for the low frequency emitter. The reason is that when the frequency is low, its changing rate should be lower. For this reason, the measuring error of the same changing rate of frequency, the geolocation performance is worse for the signal of lower frequency.
Choose the true emitter location as $\left(137.5^{\circ} \mathrm{E}, 25^{\circ} \mathrm{N}\right)$, assume the signal frequency $f_{0}=1.3 \mathrm{GHz}$, and then analyze the influence on geolocation accuracy of different frequency measurement errors. This is shown by Table 6.2, which, as found in expression (6.29), indicates that the increase of RMS of the changing rate error of frequency is in proportion to the geolocation error.

### 6.4 Single-Satellite Single-Antenna TOA-Only Geolocation

### 6.4.1 Model and Method of TOA-Only Geolocation

For a pulse signal with a fixed pulse repetition frequency (PRF), the Doppler effect of the satellite to the emitter will definitely cause a change of PRF, which is shown by the fact that the TOA

Table 6.2 CRLB of the geolocation error at a certain point with different accuracy of frequency changing rates

| RMS of changing rate error <br> of frequency $(\mathrm{Hz} / \mathrm{s})$ | 1 | 10 | 100 |
| :--- | :---: | :---: | :---: |
| Geolocation error $(\mathrm{km})$ | 0.6926 | 6.926 | 69.26 |

of each pulse intercepted by the satellite is no longer with the same interval. Changing information of pulse TOA series also implies information of the emitter position which, therefore, can be localized.
The first step in an analysis of the TOA change of pulse signal is to build a model of the PRI for it. We assume that the radar pulse signal with duration $\Delta t$ from the target emitter is received by the satellite. The signal duration contains $n$ pulses with a signal PRI $T_{r}$ and pulse width $\tau \ll T_{r}$, as shown in the following function:

$$
p(t)= \begin{cases}1, & 0<t \leq \tau  \tag{6.30}\\ 0, & \text { others }\end{cases}
$$

Then the pulse signal is

$$
\begin{equation*}
s(t)=A(t) \sum_{k=0}^{n-1} p\left(t-t_{0}-k T_{r}\right) \tag{6.31}
\end{equation*}
$$

where $t_{0}$ is the moment of the first pulse sent and $A(t)$ is the signal wave including amplitude, carrier frequency, modulation, and other factors.
Similar to the analysis of the Doppler effect of the signal frequency domain, the pulse signal periods received by the satellite are also a function of the distance between the satellite and the emitter, which is shown as

$$
\begin{equation*}
s_{r}(t)=A_{r}\left[t-\frac{r(t)}{c}\right] \sum_{k=0}^{n-1} p\left[t-t_{0}-k T_{r}-\frac{r(t)}{c}\right] \tag{6.32}
\end{equation*}
$$

where $c$ represents the signal wave propagation speed, $A_{r}[t-r(t) / c]$ means modulation of the signal received after time scaling has been considered, and $r(t)$ is the distance between the satellite and the emitter at the moment $t$.
Assume that the receiver records the arrival time when it receives the first pulse, that is, the TOA of the first pulse is $t=0$. Then the corresponding time system of the receiver can be obtained from expression (6.32), from which it is found that the value of $t_{0}$ will be equal to $(-r / c)$ where $r=r(0)$ is the distance between the satellite and the emitter at the moment of $t=0$. If $\Delta t$ is a very short time and motion acceleration is taken into account, then the second-order model for $r(t)$ at any moment in the receiving signal duration can be built as follows by using $r$, radial velocity $\dot{r}$ of the satellite to the emitter, and radial acceleration $\ddot{r}$ at the moment of $t=0$ :

$$
\begin{equation*}
r(t) \approx r+\dot{r} t+\frac{\ddot{r} t^{2}}{2} \tag{6.33}
\end{equation*}
$$

Substituting into Equation (6.32), we get

$$
\begin{equation*}
s_{r}(t) \approx A_{r}\left[\left(1-\frac{\dot{r}}{c}-\frac{\ddot{r}}{2 c} t\right) t-\frac{r}{c}\right] \sum_{k=0}^{n-1} p\left(t-T_{r, k}\right) \tag{6.34}
\end{equation*}
$$

Of course, the approximation in expression (6.34) is due to the negligence of width change of the received pulse caused by the relative theory effect. In the expression, $T_{r, k}$ is the TOA of each pulse measured by the receiver, when $\ddot{r} \neq 0$. According to the expressions (6.30) and (6.34), its expression is

$$
\begin{equation*}
T_{r, k}=\frac{\left(1-\frac{\dot{r}}{c}\right)-\sqrt{\left(1-\frac{\dot{r}}{c}\right)^{2}-\frac{2 \ddot{r}}{c} k T_{r}}}{\ddot{r} / c} \tag{6.35}
\end{equation*}
$$

It is known that the TOA of each pulse is affected by the Doppler effect and relative radial motion between the satellite and the emitter. As analyzed in Sections 6.1 to 6.3, such a relation can be used in geolocation of the emitter.
Therefore, conduct a second-order Taylor series expansion to the square root term in the numerator of expression (6.35). Then the approximate expression of $T_{r, k}$ is acquired:

$$
\begin{equation*}
T_{r, k} \approx \frac{k T_{r}}{1-\frac{\dot{r}}{c}}+\frac{\frac{\ddot{r}}{c}\left(k T_{r}\right)^{2}}{2\left(1-\frac{\dot{r}}{c}\right)^{3}} \tag{6.36}
\end{equation*}
$$

Through one observation by the satellite in a very short duration $\Delta t$, we can get a series of $T_{r, k}$ values because in the observation, $r, \dot{r}$, and $\ddot{r}$ are fixed values. Then let

$$
\left.\begin{array}{rl}
a & =1 /\left(1-\frac{\dot{r}}{c}\right)  \tag{6.37}\\
b & =\left(\frac{\ddot{r}}{c}\right) / 2\left(1-\frac{\dot{r}}{c}\right)^{3}
\end{array}\right\}
$$

and estimate PRI $T_{r}$ according to the TOA of the pulse measured. For example, get $\widehat{T}_{r}$ by subtracting TOAs of pulse series from each other and then averaging the difference obtained therefrom. Then replace $T_{r}$ in expression (6.36) with $\widehat{T}_{r}$. Expression (6.36) can become a quadratic polynomial with a constant coefficient of $k$ :

$$
\begin{equation*}
T_{r, k}=\theta_{1} k^{2}+\theta_{2} k \quad(k=0,1,2, \ldots, n) \tag{6.38}
\end{equation*}
$$

where $\theta_{1}=b \widehat{T}_{r}^{2}$ and $\theta_{2}=a \widehat{T}_{r}$. Let $\mathbf{T}_{r, k}=\left[T_{r, 0}, T_{r, 1}, \ldots, T_{r, n-1}\right]^{\mathrm{T}}, \boldsymbol{\theta}=\left[\theta_{1}, \theta_{2}\right]^{\mathrm{T}}$, and

$$
\mathbf{K}=\left[\begin{array}{ccccc}
0 & 1 & 4 & \cdots & (n-1)^{2} \\
0 & 1 & 2 & \cdots & n-1
\end{array}\right]^{\mathrm{T}}
$$

Then expression (6.38) can be written in the format of a matrix as

$$
\begin{equation*}
\mathbf{T}_{r, k}=\mathbf{K} \boldsymbol{\theta} \tag{6.39}
\end{equation*}
$$

Therefore, the least squares solution of $\boldsymbol{\theta}$ is

$$
\begin{equation*}
\widehat{\boldsymbol{\theta}}=\left(\mathbf{K}^{\mathrm{T}} \mathbf{K}\right)^{-1} \mathbf{K}^{\mathrm{T}} \mathbf{T}_{r, k} . \tag{6.40}
\end{equation*}
$$

Substitute $\hat{\boldsymbol{\theta}}$ separately into expressions $\theta_{1}$ and $\theta_{2}$ to obtain an equation system, where $a$ and $b$ can be solved to give

$$
\begin{equation*}
\hat{r}=\frac{2 \widehat{b}}{\hat{a}^{3}} c \tag{6.41}
\end{equation*}
$$

After radial acceleration is obtained, a similar geolocation solution can be performed by the single-satellite frequency measurement as introduced in Section 6.2.1, that is, $i$ represents a total $N$ times of observation by the satellite, $\mathbf{v}_{i}$ and $\mathbf{a}_{i}$ are separately the relative velocity vector and acceleration vector when the satellite is at the observation position $i, \mathbf{r}_{i}$ is the radial vector of the emitter to the satellite at the observation position $i$, and $r_{i}=\left\|\mathbf{r}_{i}\right\|$. Use geodetic coordinates to represent the position of the emitter, giving the following expression:

$$
\begin{equation*}
\hat{r}_{i}=\frac{\mathbf{v}_{i}^{\mathrm{T}} \cdot \mathbf{v}_{i}+\mathbf{r}_{i}^{\mathrm{T}} \cdot \mathbf{a}_{i}}{r_{i}}-\frac{\left(\mathbf{r}_{i}^{\mathrm{T}} \cdot \mathbf{v}_{i}\right)^{2}}{r_{i}^{3}}+\xi_{i}=g_{i}(L, B)+\xi_{i} \tag{6.42}
\end{equation*}
$$

In this expression, $\xi_{i}$ is the error of the estimated radial acceleration, which can be assumed to be mutually independent of each other, and its variance is decided by the error variance of TOA measurement; the specific deduction will be shown in the next section. For a static emitter on the earth's surface, $N$ different $\widehat{r}_{i}$ can be acquired by the satellite at $N$ observation positions, where $N$ expressions of $\widehat{r}_{i}$ can be written down in the format of the matrix as follows:

$$
\begin{equation*}
\mathbf{r}_{a}=\mathbf{G}(L, B)+\zeta \tag{6.43}
\end{equation*}
$$

where $\mathbf{r}_{a}=\left[\hat{r}_{1}, \hat{r}_{2}, \ldots, \hat{r}_{N}\right]^{\mathrm{T}}, \mathbf{G}(B, L)=\left[g_{1}(L, B), \ldots, g_{i}(L, B), \ldots, g_{N}(L, B)\right]^{\mathrm{T}}$, and $\boldsymbol{\xi}=\left[\xi_{1}, \ldots, \xi_{i}, \ldots, \xi_{N}\right]^{\mathrm{T}}$.
As for the two geolocation methods mentioned in the last two sections establish a grid of the subsatellite area and decide the size of the grid according to the coverage area and the subsatellite point. Define the grid set of points as $\sum$, for each $\left(L_{k}, B_{k}\right) \in \sum$, and calculate the cost function:

$$
\begin{equation*}
J\left(L_{k}, B_{k}\right)=\left\|\mathbf{r}_{a}-\mathbf{G}_{k}\right\|^{2} \tag{6.44}
\end{equation*}
$$

The point minimizing the cost function $J\left(L_{k}, B_{k}\right)$ is the localization output point.

### 6.4.2 Analysis of the Geolocation Error

### 6.4.2.1 Analysis of the TOA Measurement Error

At present, a popular way of measuring the TOA is that take the threshold detection moment of the pulse signal as the measurement value of the TOA. Within the limits of the SNR (signal-to-noise ratio), for example, the thermal noise caused by the receiver and the transmission noise coming from the outside environment will cause distortion of the ideal waveform, creating an error of the threshold-detection moment during TOA measurement. Reference [4]
provides the information that under conditions of high SNR, using a fixed threshold and linear rising edge, the measurement error of the TOA is

$$
\begin{equation*}
\sigma_{t}=\frac{t_{R}}{\sqrt{2 S N R}} \tag{6.45}
\end{equation*}
$$

where $t_{R}$ is the rising-edge time of the pulse, that is, the duration time of the pulse amplitude from a range of 10 to $90 \%$, a typical value of $t_{R}$ being $10-100 \mathrm{~ns}$ for a wideband receiver [5].

### 6.4.2.2 CRLB of the Geolocation Error

As the existence of the TOA measurement error can give rise to deviation of the final geolocation result, we shall research the influence imposed by the measurement error of TOA on geolocation accuracy. Assume that errors of the pulse TOA measurement are independent of each other and the Gaussian distribution with zero mean and variable $\sigma_{t}{ }^{2}$. Then according to expression (6.39), we obtain its derivative as

$$
\begin{equation*}
\mathrm{d} \mathbf{T}_{r, k}=\mathbf{K} \mathrm{d} \boldsymbol{\theta} \tag{6.46}
\end{equation*}
$$

Therefore, the covariance matrix of $\boldsymbol{\theta}$ is

$$
\begin{equation*}
\mathbf{C}_{\theta}=E\left[\mathrm{~d} \boldsymbol{\theta} \mathrm{~d} \boldsymbol{\theta}^{\mathrm{T}}\right]=\left(\mathbf{K}^{\mathrm{T}} \mathbf{K}\right)^{-1} \mathbf{K}^{\mathrm{T}} \mathbf{C}_{\mathbf{T}_{r, k}} \mathbf{K}\left(\mathbf{K}^{\mathrm{T}} \mathbf{K}\right)^{-1} . \tag{6.47}
\end{equation*}
$$

According to the assumption above,

$$
\begin{equation*}
\mathbf{C}_{\mathbf{T}_{r, k}}=\operatorname{diag}\left\{\sigma_{t}^{2}, \sigma_{t}^{2}, \ldots, \sigma_{t}^{2}\right\}_{n \times n}=\sigma_{t}^{2} \mathbf{I} . \tag{6.48}
\end{equation*}
$$

Expression (6.48) is the covariance matrix of the $T_{r, k}$ measurement error and $\mathbf{I}$ is an $n \times n$ identity matrix. Simplifying Equation (6.47) gives

$$
\begin{equation*}
\mathbf{C}_{\theta}=\sigma_{t}^{2}\left(\mathbf{K}^{\mathrm{T}} \mathbf{K}\right)^{-1} \tag{6.49}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\sigma_{\theta_{1}}{ }^{2}=k_{11}^{\prime} \sigma_{t}^{2}, \tag{6.50}
\end{equation*}
$$

where $k_{11}^{\prime}$ is an element of $\left(\mathbf{K}^{\mathrm{T}} \mathbf{K}\right)^{-1}$ in the first row and the first column. According to the expression for $\mathbf{K}$, we can deduce that

$$
\begin{equation*}
k_{11}^{\prime}=\frac{120(2 n-1)}{n\left(n^{2}-1\right)\left(3 n^{3}-9 n^{2}+8 n-4\right)} . \tag{6.51}
\end{equation*}
$$

According to the expression (6.40) to obtain $b$ and if $(1-\dot{r} / c)^{3} \approx 1$, when the error is analyzed, the following approximation can be found:

$$
\begin{equation*}
\ddot{r} \approx 2 c b \tag{6.52}
\end{equation*}
$$

Get $b=\widehat{\theta}_{1} / \widehat{T}_{r}^{2}$ from the equation for $\widehat{\boldsymbol{\theta}}$ and then substitute it into expression (6.52). This gives

$$
\begin{equation*}
\ddot{r} \approx \frac{2 c}{\widehat{T}_{r}^{2}} \widehat{\theta}_{1} . \tag{6.53}
\end{equation*}
$$

To summarize, we can get the following relation between the estimated error variance of radial acceleration and the error variance of time determination:

$$
\begin{equation*}
\sigma_{\ddot{r}}^{2} \approx \frac{4 c^{2}}{\widehat{T}_{r}^{4}} \sigma_{\theta_{1}}^{2} \approx \frac{480(2 n-1) c^{2}}{n\left(n^{2}-1\right)\left(3 n^{3}-9 n^{2}+8 n-4\right) \widehat{T}_{r}^{4}} \sigma_{t}^{2} \tag{6.54}
\end{equation*}
$$

Neglect the tiny bias resulting from substitution of $T_{r}$ with $\widehat{T}_{r}$; the estimated error of radial acceleration is approximately subject to the Gaussian distribution, of which $\sigma_{\dot{\hat{V}}}{ }^{2}$ is the variance and the mean value is 0 . In fact, the error caused by the substitution of $T_{r}$ with $\widehat{T}_{r}$ mainly comes from the Doppler effect. Now assume that the bias is $\partial T$; by differentiating expression (6.53) we can obtain

$$
\begin{equation*}
\partial \ddot{r} \approx-\frac{4 c \theta_{1}}{T_{r}^{3}} \partial T . \tag{6.55}
\end{equation*}
$$

Taking the LEO satellite and the fixed emitter on earth, for example, generally the relative radial velocity between the satellite and the emitter around the subsatellite point will not exceed $10 \mathrm{~km} / \mathrm{s}$. According to expression (6.55), we get $\partial T$ and then obtain

$$
\begin{equation*}
\left|\frac{\partial \ddot{r}_{0}}{\ddot{r}_{0}}\right| \approx\left|\frac{2 \partial T}{T_{r}}\right| \approx\left|\frac{2 \dot{r}}{c-\dot{r}}\right|<0.000067 . \tag{6.56}
\end{equation*}
$$

It is thus clear that by making $\widehat{T}_{r}$ approximate to $T_{r}$, there is small bias that can be completely ignored.
After the error variance of radial acceleration has been obtained, we take the geolocation of the ground emitter as an example to analyze the CRLB of the geolocation error. We still adopt the WGS-84 earth model $[6,7]$ and neglect the effect caused by the altitude of the earth's surface, that is, $H=0$, and indicate the position coordinates of the emitter with the geodetic longitude and latitude $(L, B)$. Then we get the joint probability density function of $\widehat{r}_{1}, \widehat{r}_{2}, \ldots, \widehat{r}_{N}$ according to expression (6.42):

$$
\begin{equation*}
p(\mathbf{r} ; L, B)=\frac{1}{\left(2 \pi \sigma_{\dot{r}}^{2}\right)^{N / 2}} \exp \left\{-\frac{\sum_{i=1}^{N}\left[\widehat{r}_{i}-g_{i}(L, B)\right]^{2}}{2 \sigma_{\dot{r}}^{2}}\right\} . \tag{6.57}
\end{equation*}
$$

By differentiating its logarithm with respect to $\boldsymbol{\eta}=(L, B)^{\mathrm{T}}$, we obtain

$$
\begin{equation*}
\frac{\partial \ln p(\mathbf{Y} ; L, B)}{\partial \boldsymbol{\eta}}=\frac{1}{\sigma_{\grave{r}}^{2}}\left[\sum_{i=1}^{N}\left[\hat{r}_{i}-g_{i}(L, B)\right] \frac{\partial g_{i}(L, B)}{\partial L}, \sum_{i=1}^{N}\left[\hat{r}_{i}-g_{i}(L, B)\right] \frac{\partial g_{i}(L, B)}{\partial B}\right] \tag{6.58}
\end{equation*}
$$

This gives the Fisher information matrix

$$
\begin{equation*}
\mathbf{I}(\mathbf{X})=\frac{1}{\sigma_{\ddot{r}}^{2}} \mathbf{H}^{\mathrm{T}} \mathbf{H} \tag{6.59}
\end{equation*}
$$

where

$$
\mathbf{H}=\left[\begin{array}{llll}
\frac{\partial g_{1}(L, B)}{\partial L} & \frac{\partial g_{2}(L, B)}{\partial L} & \cdots & \frac{\partial g_{N}(L, B)}{\partial L} \\
\frac{\partial g_{1}(L, B)}{\partial B} & \frac{\partial g_{2}(L, B)}{\partial B} & \cdots & \frac{\partial g_{N}(L, B)}{\partial B}
\end{array}\right]^{\mathrm{T}}
$$

After variable transformation, the CRLB in the ECEF coordinate system is

$$
\begin{equation*}
C R L B_{(x, y, z)}=\sigma_{\stackrel{r}{r}}^{2}\left[\frac{\partial F(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}}\left(\mathbf{H}^{\mathrm{T}} \mathbf{H}\right)^{-1} \frac{\partial F(\boldsymbol{\eta})^{\mathrm{T}}}{\partial \boldsymbol{\eta}}\right]_{j j} . \tag{6.60}
\end{equation*}
$$

Suppose $c_{11}, c_{22}, c_{33}$ are elements on the diagonal line of the right matrix in the expression above; then according to the definition of geometric dilution of the geolocation error,

$$
\begin{align*}
G D O P=\sigma_{\ddot{r}} \sqrt{\sum_{i=1}^{3} c_{i i}} & \approx \frac{21.91 c \sqrt{(2 n-1)\left(c_{11}+c_{22}+c_{33}\right)}}{\sqrt{n\left(n^{2}-1\right)\left(3 n^{3}-9 n^{2}+8 n-4\right)} T_{r}^{2}} \sigma_{t} \\
& \approx \frac{21.91 t_{R} c \sqrt{(2 n-1)\left(c_{11}+c_{22}+c_{33}\right)}}{\sqrt{2 \operatorname{SNR} \cdot n\left(n^{2}-1\right)\left(3 n^{3}-9 n^{2}+8 n-4\right)} T_{r}^{2}} \tag{6.61}
\end{align*}
$$

### 6.4.3 Geolocation Simulation

Assuming that the rising-edge time of a radar pulse is 20 ns and the SNR of the received signal is up to 13 dB , then according to expression (6.45), the RMS of the TOA measurement error $\sigma_{t}=3.166 \mathrm{~ns}$. Suppose the TOA error of each pulse measurement is the zero-mean Gaussian noise, which is independent of each other; we then carry out calculations according to the deduction in Section 6.4.2. Assuming the total observation time is 0.6 second, then $n=465$. $\widehat{T}_{r}$ is the actual PRI, which equals $1.29 \times 10^{-3}$ seconds. This is then substitute into expression (6.54) to get the RMS of the estimated error of radial acceleration, which is about $2.2 \mathrm{~m} / \mathrm{s}^{2}$.

After the estimated error of radial acceleration has been obtained, the GDOP graph of the same area as mentioned Section 6.3 can be acquired through calculation, which is shown in Figure 6.5.
The essence of the TOA-only passive geolocation method and the FCR measured geolocation method is to estimate the radial acceleration in order to carry out geolocation by measuring signal parameters. They have similar GDOP graphs. Only because of different parameter measurement errors do they have different performances of geolocation.

### 6.5 Single-Satellite Interferometer Phase Rate of Changing-Only Geolocation

### 6.5.1 Geolocation Model

Using two antennas consisting of an interferometer carried by a satellite, the information of the phase difference rate of changing $\dot{\phi}(t)$ of the transmitted signal from the emitter at an unknown position can be measured, where $\dot{\phi}(t)$ implies the position information of the emitter.


Figure 6.5 Ground GDOP graph for the single-satellite TOA-only measured passive geolocation

Assume in the system $\{n\}$ for the NED (north-east-down) coordinate system that the vectors of the emitter position and the observer position are separately $\mathbf{x}_{T, n}$ and $\mathbf{x}_{O, n}$. Then the position and velocity of the observer can be obtained from a navigation system such as the GPS (global positioning system). Suppose the yaw angle $\psi$, pitch angle $\theta$, and roll angle $\varphi$ are output by the attitude sensor of the observer; then definitions of the three angles are as shown in Section 2.3.1.
Assume that the ' $3-2-1$ ' Euler angles attitude rotation order is adopted, that is, $\varphi$ rolls along the $x$ axis first, then $\theta$ rolls along the $y$ axis, and $\psi$ rolls along the $z$ axis last. In System $\{b\}$ of the body coordinates system, the position vector of the emitter is

$$
\begin{equation*}
\mathbf{x}_{T, b}=\mathbf{A}_{n 2 b}\left(\mathbf{x}_{T, n}-\mathbf{x}_{O, n}\right), \tag{6.62}
\end{equation*}
$$

where

$$
\begin{aligned}
& \mathbf{A}_{n 2 b}=\mathbf{R}_{x}^{\mathrm{T}}(\varphi) \mathbf{R}_{y}^{\mathrm{T}}(\theta) \mathbf{R}_{z}^{\mathrm{T}}(\psi), \\
& \mathbf{R}_{x}(\theta)=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{array}\right], \mathbf{R}_{y}(\theta)=\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right], \\
& \mathbf{R}_{z}(\theta)=\left[\begin{array}{ccc}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right] .
\end{aligned}
$$

Assume two receivers are installed on the satellite. The interferometer vector in System $\{b\}$ of the body axis system of the receivers is $\mathbf{B}_{b}$ and the interferometer baseline vector $\mathbf{B}_{b}$ in System $\{n\}$ is

$$
\begin{equation*}
\mathbf{B}_{n}=\mathbf{A}_{b 2 n} \mathbf{B}_{b} . \tag{6.63}
\end{equation*}
$$

Then the phase difference received by the interferometer is

$$
\begin{equation*}
\phi(t)=\left(2 \pi \frac{\mathbf{B}_{n}^{\mathrm{T}} \cdot \mathbf{x}_{n}}{\lambda\left\|\mathbf{x}_{n}\right\|}+\phi_{0}\right) \bmod 2 \pi, \tag{6.64}
\end{equation*}
$$

where $\lambda$ is the signal wavelength, $\phi_{0}$ is a fixed phase bias brought by the amplitude or phase inconsistency between the two interferometer channels, and $\mathbf{x}_{n}=\mathbf{x}_{T, n}-\mathbf{x}_{O, n}$. For the convenience of description, define the unit vector in the direction of the emitter as

$$
\begin{equation*}
\mathbf{u}_{n} \triangleq \frac{\mathbf{x}_{n}}{\left\|\mathbf{x}_{n}\right\|} \tag{6.65}
\end{equation*}
$$

Therefore, expression (6.64) can be transformed into the following format:

$$
\begin{equation*}
\phi(t)=\left(\frac{2 \pi}{\lambda} \mathbf{B}_{n}^{\mathrm{T}} \cdot \mathbf{u}_{n}+\phi_{0}\right) \bmod 2 \pi \tag{6.66}
\end{equation*}
$$

Differentiating it with respect to the expression, we can get

$$
\begin{equation*}
\dot{\phi}(t)=\frac{2 \pi}{\lambda}\left(\frac{\mathrm{~d} \mathbf{B}_{n}^{\mathrm{T}}}{\mathrm{~d} t} \cdot \mathbf{u}_{n}+\mathbf{B}_{n}^{\mathrm{T}} \cdot \frac{\mathrm{~d} \mathbf{u}_{n}}{\mathrm{~d} t}\right) . \tag{6.67}
\end{equation*}
$$

Changing the rate of the interferometer baseline vector in the process of rotation gives

$$
\begin{align*}
\frac{\mathrm{d} \mathbf{B}_{n}^{\mathrm{T}}}{\mathrm{~d} t} & =\mathbf{B}_{b}^{T} \frac{\mathrm{~d} \mathbf{A}_{b 2 n}^{\mathrm{T}}}{\mathrm{~d} t} \\
& =\mathbf{B}_{b}^{\mathrm{T}}\left(\frac{\mathrm{~d} \mathbf{A}_{b 2 n}}{\mathrm{~d} \psi} \dot{\psi}+\frac{\mathrm{d} \mathbf{A}_{b 2 n}}{\mathrm{~d} \theta} \dot{\theta}+\frac{\mathrm{d} \mathbf{A}_{b 2 n}}{\mathrm{~d} \varphi} \dot{\varphi}\right)^{\mathrm{T}} \tag{6.68}
\end{align*}
$$

where $\dot{\psi}, \dot{\theta}$, and $\dot{\varphi}$ are 3D attitude changing rates output by the gyroscope. Define

$$
\dot{\mathbf{A}}_{b 2 n}=\left[\frac{\partial \mathbf{A}_{b 2 n}^{\mathrm{T}}}{\partial \psi} \frac{\partial \mathbf{A}_{b 2 n}^{\mathrm{T}}}{\partial \theta} \frac{\partial \mathbf{A}_{b 2 n}^{\mathrm{T}}}{\partial \varphi}\right]^{\mathrm{T}} \text { and } \dot{\mathbf{A}}=\left[\begin{array}{llll}
\dot{\psi} \mathbf{I}_{3} & \dot{\theta} \mathbf{I}_{3} & \dot{\varphi} \mathbf{I}_{3}
\end{array}\right]^{\mathrm{T}}
$$

The above expressions can now be written in matrix form:

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{B}_{n}^{\mathrm{T}}}{\mathrm{~d} t}=\mathbf{B}_{b}^{\mathrm{T}} \dot{\mathbf{A}}^{\mathrm{T}} \dot{\mathbf{A}}_{b 2 n} \tag{6.69}
\end{equation*}
$$

The derivative of expression (6.65) can be found as

$$
\begin{align*}
\frac{\mathrm{d} \mathbf{u}_{n}}{\mathrm{~d} t} & =\mathrm{d}\left(\frac{1}{\left\|\mathbf{x}_{n}\right\|}\right) / \mathrm{d} t \cdot \mathbf{x}_{n}+\frac{\dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|}=-\frac{\mathbf{x}_{n}^{\mathrm{T}} \dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|^{3}} \mathbf{x}_{n}+\frac{\dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|} \\
& =-\frac{\mathbf{x}_{n} \mathbf{x}_{n}^{\mathrm{T}} \dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|^{3}}+\frac{\dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|} \\
& =\frac{\left(\mathbf{I}_{3}-\frac{\mathbf{x}_{n} \mathbf{x}_{n}^{\mathrm{T}}}{\left\|\mathbf{x}_{n}\right\|^{2}}\right) \dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|}=\frac{\left(\mathbf{I}_{3}-\mathbf{u}_{n} \mathbf{u}_{n}^{\mathrm{T}}\right) \dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|} \tag{6.70}
\end{align*}
$$

Substituting this into expression (6.67) gives

$$
\begin{equation*}
\dot{\phi}(t)=\frac{2 \pi}{\lambda} \mathbf{B}_{b}^{\mathrm{T}}\left(\dot{\mathbf{A}}^{\mathrm{T}} \dot{\mathbf{A}}_{b 2 n} \cdot \mathbf{u}_{n}+\mathbf{A}_{b 2 n}^{\mathrm{T}} \cdot \frac{\left(\mathbf{I}-\mathbf{u}_{n} \mathbf{u}_{n}^{\mathrm{T}}\right) \dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|}\right) \tag{6.71}
\end{equation*}
$$

### 6.5.2 Geolocation Algorithm

Estimate the emitter position to be $\mathbf{x}_{t}=\left[\begin{array}{ll}L & B\end{array}\right]^{\mathrm{T}}$. Then $\dot{\phi}_{n m}$ observed at the $n$th moment with the emitter position is

$$
\begin{equation*}
\dot{\phi}_{n m}=f\left(\mathbf{x}_{t}\right)+v_{n} \quad(n=1, \ldots, N) . \tag{6.72}
\end{equation*}
$$

Suppose that the measurement errors $v_{n}$ are not coherent with each other and comply with the Gaussian distribution of $N\left(0, \sigma_{\dot{\phi}}^{2}\right)$. As $\dot{\phi}_{n m}$ is the nonlinear function of the emitter position $\mathbf{x}_{t}$, various methods can be adopted to carry out estimations of geolocation, such as grid search, Newton iteration, nonlinear least squares, extended Kalman filter, and other nonlinear filtering methods. However, as the calculation burden is heavy and geolocation accuracy is also affected by the size of the grid, the grid search method cannot be very effective. The initial value is indispensable in calculations using the Newton iteration, nonlinear least squares, extended Kalman filter, and other methods. A 'bad' choice of the initial value leads to 'bad' stability of the algorithm. As the geolocation process is by nature a process with nonlinear optimization under constraint, we will consider using the particle filter method here.
Assume the position of the $i$ particle at the $n$th moment is $\mathbf{x}_{n, i}$, the probability weighting value is $\omega_{n, i}$, and they are expressed as $\left(\mathbf{x}_{n, i}, \omega_{n, i}\right)$. The probability weighting value at the initial zero moment ( $\mathbf{x}_{0, i}, \omega_{0, i}$ ) meets the probability density distribution:

$$
\left.\begin{array}{l}
\mathbf{x}_{0, i} \rightarrow p_{0}(\mathbf{x})  \tag{6.73}\\
\omega_{0, i}=p_{0}\left(\mathbf{x}_{0, i}\right) \quad(i=1, \ldots, I)
\end{array}\right\}
$$

Without prior information, we can suppose that the initial particles ( $\mathbf{x}_{0, i}, \omega_{0, i}$ ) present a uniform distribution in the reconnaissance covering area of the satellite:

$$
\begin{equation*}
p_{0}\left(\mathbf{x}_{0, i}\right)=1 / I . \tag{6.74}
\end{equation*}
$$

Suppose that $n$ observation values are obtained. Let $\dot{\boldsymbol{\varphi}}_{m}^{n}=\left[\dot{\phi}_{1}, \ldots, \dot{\phi}_{n}\right]$; then the posterior probability for each corresponding particle is

$$
\begin{align*}
\omega_{n, i} & =p\left(\mathbf{x}_{n, i} \mid \dot{\boldsymbol{\varphi}}_{m}^{n}\right)=\prod_{l=1}^{n} c_{l} \exp \left\{-\frac{\left(\phi_{n m}-f\left(\mathbf{x}_{n, i}\right)^{2}\right.}{2 \sigma_{\dot{\phi}}^{2}}\right\} \\
& =\frac{1}{c_{n}} \exp \left\{-\sum_{l=1}^{n} \frac{\left(\phi_{n m}-f\left(\mathbf{x}_{n, i}\right)^{2}\right.}{2 \sigma_{\dot{\phi}}^{2}}\right\} \tag{6.75}
\end{align*}
$$

where $c_{n}$ is the normalization coefficient with

$$
c_{n}=\sum_{i=1}^{I} \exp \left\{-\sum_{l=1}^{n} \frac{\left(\phi_{n m}-f\left(\mathbf{x}_{n, i}\right)^{2}\right.}{2 \sigma_{\phi}^{2}}\right\}
$$

The estimated output at the final $n$th moment is

$$
\begin{equation*}
\widehat{\mathbf{x}}_{t, n}=\sum_{i=1}^{I} \omega_{n, i} \mathbf{x}_{n, i} \tag{6.76}
\end{equation*}
$$

Here the particle distribution at the current $n$th moment will be used to produce the particle distribution at the $(n+1)$ th moment, which is called the importance resampling technique. Resampling contains many methods. This article uses a method called Gaussian resampling, that is, calculating the weighted secondary moment of the current particle:

$$
\begin{equation*}
\mathbf{P}_{n}=\sum_{i=1}^{I} \omega_{n, i}\left(\mathbf{x}_{n, i}-\widehat{\mathbf{x}}_{t, n}\right)\left(\mathbf{x}_{n, i}-\widehat{\mathbf{x}}_{t, n}\right)^{\mathrm{T}} \tag{6.77}
\end{equation*}
$$

Using Cholesky's method of decomposing the secondary moment, we get

$$
\begin{equation*}
\mathbf{P}_{n}=\mathbf{R}_{n}^{\mathrm{T}} \mathbf{R}_{n} \tag{6.78}
\end{equation*}
$$

Let the random vector $\mathbf{y}_{i} \rightarrow N(0, \mathbf{I})$, where $\mathbf{I}$ is the identity matrix. It is clear that the particle at the $(n+1)$ th moment is

$$
\begin{equation*}
\mathbf{x}_{n+1, i}=\widehat{\mathbf{x}}_{t, n}+\mathbf{R}_{n}^{\mathrm{T}} \mathbf{y}_{i} \tag{6.79}
\end{equation*}
$$

Obviously, according to expressions (6.78) and (6.79), we know that

$$
\begin{equation*}
E\left[\left(\mathbf{x}_{n+1, i}-\widehat{\mathbf{x}}_{t, n}\right)\left(\mathbf{x}_{n+1, i}^{\mathrm{T}}-\widehat{\mathbf{x}}_{t, n}\right)\right]=E\left[\mathbf{R}_{n}^{\mathrm{T}} \mathbf{y}_{i} \mathbf{y}_{i} \mathbf{R}_{n}\right]=\mathbf{R}_{n}^{\mathrm{T}} E\left[\mathbf{y}_{i} \mathbf{y}_{i}^{\mathrm{T}}\right] \mathbf{R}_{n}=\mathbf{P}_{n} \tag{6.80}
\end{equation*}
$$

The expression above shows that $\mathbf{x}_{n+1, i}$ still remains the weighted variance Gaussian distribution obtained by calculation at the $n$th moment, which is consistent with the Gaussian distribution of $N\left(\mathbf{x}_{t, n}, \mathbf{P}_{n}\right)$.

### 6.5.3 CRLB of the Geolocation Error

Let

$$
\mathbf{G}\left(\mathbf{x}_{n}\right)=\frac{\left(\mathbf{I}-\mathbf{u}_{n} \mathbf{u}_{n}^{\mathrm{T}}\right) \dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|}
$$

Then the phase rate of changing can be written as

$$
\begin{equation*}
\dot{\phi}(t)=\frac{2 \pi}{\lambda} \mathbf{B}_{b}^{\mathrm{T}}\left(\dot{\mathbf{A}}^{\mathrm{T}} \dot{\mathbf{A}}_{b 2 n} \cdot \mathbf{u}_{n}+\mathbf{A}_{b 2 n}^{\mathrm{T}} \cdot \mathbf{G}\left(\mathbf{x}_{n}\right)\right) \tag{6.81}
\end{equation*}
$$

By differentiating with respect to the emitter position, we get

$$
\begin{equation*}
\frac{\partial \dot{\phi}}{\partial \mathbf{x}_{T, n}}=\frac{2 \pi}{\lambda} \mathbf{B}_{b}^{\mathrm{T}}\left(\dot{\mathbf{A}}^{\mathrm{T}} \dot{\mathbf{A}}_{b 2 n} \cdot \frac{\partial \mathbf{u}_{n}}{\partial \mathbf{x}_{T, n}}+\mathbf{A}_{b 2 n}^{\mathrm{T}} \cdot \frac{\partial \mathbf{G}\left(\mathbf{x}_{n}\right)}{\partial \mathbf{x}_{n}}\right) \tag{6.82}
\end{equation*}
$$

The definition of each derivative is shown as follows:

$$
\frac{\partial \mathbf{u}_{n}}{\partial \mathbf{x}_{T, n}}=\frac{\partial \mathbf{u}_{n}}{\partial \mathbf{x}_{n}}=\mathbf{x}_{n} \mathrm{~d}\left(\frac{1}{\left\|\mathbf{x}_{n}\right\|}\right) / \mathrm{d} \mathbf{x}_{n}+\frac{\mathbf{I}_{3}}{\left\|\mathbf{x}_{n}\right\|}=\frac{\left\|\mathbf{x}_{n}\right\|^{2} \mathbf{I}_{3}-\mathbf{x}_{n} \mathbf{x}_{n}^{\mathrm{T}}}{\left\|\mathbf{x}_{n}\right\|^{3}}
$$

and

$$
\begin{aligned}
\frac{\partial \mathbf{G}\left(\mathbf{x}_{n}\right)}{\partial \mathbf{x}_{n}} & =\frac{\partial\left(\frac{\dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|}-\frac{\mathbf{x}_{n} \mathbf{x}_{n}^{\mathrm{T}} \dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|^{3}}\right)}{\partial \mathbf{x}_{n}}=\frac{\partial\left(\frac{\dot{\mathbf{x}}_{n}}{\left\|\mathbf{x}_{n}\right\|}-\frac{\left(\mathbf{x}_{n}^{\mathrm{T}} \dot{\mathbf{x}}_{n}\right) \mathbf{x}_{n}}{\left\|\mathbf{x}_{n}\right\|^{3}}\right)}{\partial \mathbf{x}_{n}} \\
& =-\frac{\dot{\mathbf{x}}_{n} \mathbf{x}_{n}^{\mathrm{T}}}{\left\|\mathbf{x}_{n}\right\|^{3}}-\left[-3 \frac{\left(\mathbf{x}_{n}^{\mathrm{T}} \dot{\mathbf{x}}_{n}\right) \mathbf{x}_{n} \mathbf{x}_{n}^{\mathrm{T}}}{\left\|\mathbf{x}_{n}\right\|^{5}}+\frac{\mathbf{x}_{n} \dot{\mathbf{x}}_{n}^{\mathrm{T}}+\mathbf{x}_{n}^{\mathrm{T}} \dot{\mathbf{x}}_{n} \mathbf{I}_{3}}{\left\|\mathbf{x}_{n}\right\|^{3}}\right] \\
& =3 \frac{\left(\mathbf{x}_{n}^{\mathrm{T}} \dot{\mathbf{x}}_{n}\right) \mathbf{x}_{n} \mathbf{x}_{n}^{\mathrm{T}}}{\left\|\mathbf{x}_{n}\right\|^{5}}-\frac{\dot{\mathbf{x}}_{n} \mathbf{x}_{n}^{\mathrm{T}}+\dot{\mathbf{x}}_{n}^{\mathrm{T}} \mathbf{x}_{n}+\mathbf{x}_{n}^{\mathrm{T}} \dot{\mathbf{x}}_{n} \mathbf{I}_{3}}{\left\|\mathbf{x}_{n}\right\|^{3}}
\end{aligned}
$$

Then the Fisher information matrix is obtained as follows:

$$
\begin{equation*}
\mathbf{J}=\frac{1}{\sigma_{\phi}^{2}} \mathbf{H}^{\mathrm{T}} \mathbf{H}, \tag{6.83}
\end{equation*}
$$

where

$$
\mathbf{H}=\left(\frac{\partial \dot{\phi}_{1}}{\partial \mathbf{x}_{T}} \cdots \frac{\partial \dot{\phi}_{N}}{\partial \mathbf{x}_{T}}\right)^{\mathrm{T}} .
$$

According to reference [2], we obtain the CRLB under the constraint condition as

$$
\begin{equation*}
\text { CRLB }=\mathbf{J}^{-1}-\mathbf{J}^{-1} \mathbf{F}\left(\mathbf{F}^{\mathrm{T}} \mathbf{J}^{-1} \mathbf{F}\right)^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{J}^{-1} \tag{6.84}
\end{equation*}
$$

In this expression, $\mathbf{F}$ is the gradient vector of the constraint equation. If the regular spherical constraint of the earth of $\mathbf{x}_{T}^{\mathrm{T}} \mathbf{x}_{T}=R^{2}$ is met, it is clear that $\mathbf{F}=\mathbf{x}_{T}^{\mathrm{T}}$.

### 6.5.4 Calculation Analysis of the Geolocation Error

Assume that the signal frequency is 1 GHz , the baseline length is 10 m , the rotational velocity of the interferometer baseline is 10 seconds per round, the total observation time is 10 seconds, and the observation interval is 0.1 second. As the relative relationship between the antenna rotation plane and the satellite orbital plane is different, we can get the geometric distribution of the geolocation error CRLB under three different conditions as follows:

1. Case 1. Antenna rotation plane parallel to the satellite orbital plane

When the antenna rotation plane is parallel to the satellite orbital plane, the phase difference curve of the emitter on the earth's surface at the subsatellite point and the phase rate of the changing curve are obtained as shown in Figure 6.6a and b. The GDOP of the geolocation error CRLB of the emitter at different positions in the area near the subsatellite point is shown in Figure 6.6c.

It can be seen from Figure 6.6a and $b$ that both the phase difference curve and the phase rate of the changing curve comply with the rule of sine curve. From Figure 6.6 c we can see that there is an unobservable strip of area along the flight direction of the satellite subsatellite track, where for the geolocation error of the area on both sides, the CRLB can be within kilometers.


Figure 6.6 Simulation result under the condition that the antenna rotation plane is parallel to the satellite orbital plane. (a) Phase difference curve of the emitter at the subsatellite point, (b) phase rate of the changing curve of the emitter at the subsatellite point, and (c) contour map of the geolocation error near the subsatellite point
2. Case 2. Antenna rotation plane perpendicular to the velocity vector of the satellite at the initial moment

When the antenna rotation plane is perpendicular to the velocity vector of the satellite at the initial moment, the phase difference curve of the emitter on the earth's surface at the subsatellite point and the phase rate of the changing curve are obtained as shown in Figure 6.7a and b. The GDOP of the geolocation error CRLB of the emitter at different ground positions in the area near the subsatellite point is shown in Figure 6.7c.

It can be seen from Figure 6.7a and $b$ that both the phase difference curve and the phase rate of the changing curve comply with the rule of sine curve. From Figure 6.7 c we can see that there is an unobservable strip of area along the ground projection direction of the rotating satellite antenna, where for the geolocation error for both sides of subsatellites the CRLB can be within kilometers.


Figure 6.7 Simulation results under the condition that the antenna rotation plane is perpendicular to the velocity vector of the satellite at the initial moment. (a) Phase difference curve of the emitter at the subsatellite point, (b) phase rate of the changing curve of the emitter at the subsatellite point, and (c) contour map of the geolocation error near the subsatellite point
3. Case 3. Antenna rotation plane perpendicular to the position vector of the satellite at the initial moment

When the antenna rotation plane is perpendicular to the position vector of the satellite (the starting point of the vector is the earth center and the ending point is the position) at the initial moment, the phase difference curve of the emitter on the earth's surface at the subsatellite point and the phase rate of the changing curve are obtained as shown in Figure 6.8a and b . The GDOP of the geolocation error CRLB of the emitter at different positions in the area near the subsatellite point is shown in Figure 6.8c.

It can be seen from Figure 6.8a and b that the phase difference curve and the phase rate of the changing curve still comply with the rule of sine curve. From Figure 6.8c, we can see that, in this case, for the geolocation error of the emitter in the large area near the subsatellite point, the CRLB can be within kilometers. There is no unobservable


Figure 6.8 Simulation result under the condition that the antenna rotation plane is perpendicular to the position vector of the satellite at the initial moment. (a) Phase difference curve of the emitter at the subsatellite point, (b) phase rate of the changing curve of the emitter at the subsatellite point, and (c) contour map of the geolocation error near the subsatellite point
strip of area, as in Cases 1 and 2, and the distribution of the geolocation error is comparatively ideal.

From the analysis above, geolocation by measuring the phase rate of changing the long baseline interferometer (LBI) through the rotation motion of the satellite, for the emitter on earth, will adopt the mode in which the antenna rotation plane of the interferometer is perpendicular to the position vector of the satellite at the initial moment. The benefit of the mode is a comparatively ideal distribution of the geolocation error, in which case geolocation accuracy below 1 km in a large coverage area can be achieved.

## References

1. Lu, A. and Kong, X. (2004) Passive localization by single satellite frequency measurements. Journal on Communications, 25(9): 160-168 (in Chinese).
2. Kay, S. (1998) Fundamentals of Statistical Signal Processing, Volume I: Estimation Theory/Volume II: Detection Theory, Prentice Hall PTR.
3. Sun, Z. (2001) Passive localization technology based on kinematics. Guidance and Fuze, 22(1): 40-44 (in Chinese).
4. Li, Z. (2003) Research on Technology of Passive Localization and Tracking for Dynamic Emitter by a Non-Maneuvering Single Observer. Changsha: Graduate School of National University of Defense Technology (in Chinese).
5. Zhou, Y., An, W., Guo, F., et al. (2009) Principles of Electronic Warfare. Beijing: Publishing House of Electronics Industry (in Chinese).
6. Liu, L. (1992) Orbital Dynamics of Manned Satellite. Beijing: China Higher Education Press (in Chinese).
7. Liu, L. (2000) Orbit Theory of Spacecraft. Beijing: National Defence Industry Press (in Chinese).

## 7

## Geolocation by Near-Space Platforms

### 7.1 An Overview of Geolocation by Near-Space Platforms

### 7.1.1 Near-Space Platform Overview

'Near-space' is also known as 'near aerospace,' 'terrestrial space,' or 'transition zone between aerospace and atmosphere' [1-3]. In general, the aerospace is defined as the space over 100 km above the earth's surface, for which the spacecraft could reach, and the aerial space is the space below 20 km above the earth's surface, for which the aircraft could reach, as shown in Figure 7.1. However, as a widely accepted concept at present, the 'near-space' mentioned here is defined as the space from the altitude about 20 km (which is close to the internationally recognized upper limit of the controlled airspace) to 100 km (which is close to the internationally recognized lower limit of the aerospace). Therefore, 'near-space' can be briefly considered as the airspace from the highest flight altitude of existing aircraft (about 20 km ) to the lowest altitude of satellite orbits (about 100 km ) [3].
Since the near-space flight vehicle first appeared in Schriever-III space warfare exercises by the US army, the near-space flight vehicle has aroused widespread concern around the world. Especially in the United States, a number of positive discussions and researches were developed with respect to the correlated techniques, functions, and military applications. In their opinion, the near-space is a space environment with low risks and high returns. The military near-space fight vehicle is rarely affected by weather conditions, so the risk of the flight vehicle being attacked by a ground station is usually lower than that of an aircraft. Because the near-space flight vehicle is closer to the earth than the LEO (low earth orbit) satellite, as a result it can provide more accurate information than satellites. As mentioned above, the near-space reconnaissance application is a promising research area.
Viewed from the current development, the near-space flight vehicles mainly include the free-floating balloon, airship, unmanned aircraft, hypersonic flight vehicle, and aerospace plane [1-3]. Owing to the fact thatthe near-space flight vehicle is superior to aircraft and satellite platforms in electronic reconnaissance applications, it has great application potential.

[^2]

Figure 7.1 Near-space view

### 7.1.2 Geolocation by the Near-Space Platform

Due to the characteristic of the near-space platform having a 'large coverage area, it has a potential for detecting distant emitters in a large area by comparing them with the ground-based or airborne electronic reconnaissance platforms. Since the features and speed of the free-floating balloon, airship, unmanned aircraft, hypersonic flight vehicle, and aerospace plane are different from each other, the geolocation method of a near-space platform electronic reconnaissance system is closely related to the type of platform.

For the free-floating balloon and airship, which have a rather slow moving speed, the single-platform LOS (line-of-sight) geolocation, multistation triangulation, or the TDOA (time difference of arrival) geolocation algorithm can be used. For high-altitude unmanned aerial vehicles (UAVs) or hypersonic flight vehicles, geolocation for a fixed or moving emitter on the earth's surface may be achieved by utilizing the platform's movement, combined with the single-station passive geolocation theories based on particle kinematics.

### 7.2 Multiplatform Triangulation

### 7.2.1 Theory of $2 D$ Triangulation

Assuming that two observers are located at $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$, the emitter with an unknown position is at $(x, y)$, and two measured angles are $\theta_{1}$ and $\theta_{2}$ respectively, the two LOS from two different directions will meet at the point where the emitter is located, as shown in Figure 7.2.


Figure 7.2 The dual-station triangulation sketch

Based on the definition of an angle, we find that

$$
\left.\begin{array}{l}
\operatorname{tg} \theta_{1}=\frac{y-y_{1}}{x-x_{1}} \\
\operatorname{tg} \theta_{2}=\frac{y-y_{2}}{x-x_{2}} \tag{7.1}
\end{array}\right\}
$$

This may be written as

$$
\left.\begin{array}{l}
\left(x-x_{1}\right) \operatorname{tg} \theta_{1}=y-y_{1}  \tag{7.2}\\
\left(x-x_{2}\right) \\
\operatorname{tg} \theta_{2}=y-y_{2}
\end{array}\right\}
$$

or in a matrix form as

$$
\begin{equation*}
\mathbf{A X}=\mathbf{Z} \tag{7.3}
\end{equation*}
$$

where

$$
\mathbf{A}=\left[\begin{array}{ll}
-\operatorname{tg} \theta_{1} & 1 \\
-\operatorname{tg} \theta_{2} & 1
\end{array}\right], \mathbf{X}=\left[\begin{array}{l}
x \\
y
\end{array}\right], \mathbf{Z}=\left[\begin{array}{l}
-x_{1} \operatorname{tg} \theta_{1}+y_{1} \\
-x_{2} \operatorname{tg} \theta_{2}+y_{2}
\end{array}\right] .
$$

From these we can obtain

$$
\mathbf{X}=\mathbf{A}^{-1} \mathbf{Z}=\frac{1}{\operatorname{tg} \theta_{2}-\operatorname{tg} \theta_{1}}\left[\begin{array}{c}
x_{2} \operatorname{tg} \theta_{2}-x_{1} \operatorname{tg} \theta_{1}-y_{2}+y_{1} \\
x_{2} \operatorname{tg} \theta_{1} \operatorname{tg} \theta_{2}-x_{1} \operatorname{tg} \theta_{1} \operatorname{tg} \theta_{2}-y_{2} \operatorname{tg} \theta_{1}+y_{1} \operatorname{tg} \theta_{2}
\end{array}\right] .
$$

If there are more observers, this can be solved by expanding the expression (7.2) to more rows and using the least squares method [4].

### 7.2.2 Error Analysis for Dual-Station Triangulation

Since errors will exist when measuring $\theta_{1}$ and $\theta_{2}$, we assume that the two measured angle errors are $\delta \theta_{1}$ and $\delta \theta_{2}$, corresponding to $\theta_{1}$ and $\theta_{2}$ respectively. The measuring angle errors can be divided into a random error and bias, where the bias may be calibrated. Therefore the random error will have a significant influence on geolocation. In the analysis mentioned below, it is assumed that the measuring angle errors follow the Gaussian distribution with zero mean and the measuring errors for the two stations are independent of each other.

In order to analyze the error of triangulation, by finding the partial derivative of expression (7.2) and making a proper rearrangement, we can obtain

$$
\left.\begin{array}{l}
-\delta x \operatorname{tg} \theta_{1}+\delta y=\left(x-x_{1}\right) \sec ^{2} \theta_{1} \delta \theta_{1}  \tag{7.4}\\
-\delta x \operatorname{tg} \theta_{2}+\delta y=\left(x-x_{2}\right) \sec ^{2} \theta_{2} \delta \theta_{2}
\end{array}\right\},
$$

or in a matrix form as

$$
\begin{equation*}
\mathbf{A} \delta \mathbf{X}=\mathbf{B} \tag{7.5}
\end{equation*}
$$

where

$$
\delta \mathbf{X}=\left[\begin{array}{l}
\delta x \\
\delta y
\end{array}\right], \mathbf{B}=\left[\begin{array}{c}
\left(x-x_{1}\right) \sec ^{2} \theta_{1} \delta \theta_{1} \\
\left(x-x_{2}\right) \sec ^{2} \theta_{2} \delta \theta_{2}
\end{array}\right] .
$$

We can the obtain

$$
\begin{equation*}
\delta \mathbf{X}=\mathbf{A}^{-1} \mathbf{B} \triangle \mathbf{T} \cdot \mathbf{B} \tag{7.6}
\end{equation*}
$$

Let

$$
\mathbf{T}=\left[\begin{array}{ll}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{array}\right] \text { and } \mathbf{B}=\left[\begin{array}{l}
B_{1} \delta \theta_{1} \\
B_{2} \delta \theta_{2}
\end{array}\right]
$$

The above expression may then be written as

$$
\delta \mathbf{X}=\left[\begin{array}{l}
T_{11} B_{1} \delta \theta_{1}+T_{12} B_{2} \delta \theta_{2}  \tag{7.7}\\
T_{21} B_{1} \delta \theta_{1}+T_{22} B_{2} \delta \theta_{2}
\end{array}\right]
$$

We can obtain the covariance matrix as

$$
\mathbf{P}=E\left[\delta \mathbf{X} \delta \mathbf{X}^{\mathrm{T}}\right]=\left[\begin{array}{cc}
T_{11}^{2} B_{1}^{2} \sigma_{\theta 1}^{2}+T_{12}^{2} B_{2}^{2} \sigma_{\theta 2}^{2} & T_{11} T_{21} B_{1}^{2} \sigma_{\theta 1}^{2}+T_{12} T_{22} B_{2}^{2} \sigma_{\theta 2}^{2}  \tag{7.8}\\
T_{11} T_{21} B_{1}^{2} \sigma_{\theta 1}^{2}+T_{12} T_{22} B_{2}^{2} \sigma_{\theta 2}^{2} & T_{21}^{2} B_{1}^{2} \sigma_{\theta 1}^{2}+T_{22}^{2} B_{2}^{2} \sigma_{\theta 2}^{2}
\end{array}\right]
$$

Therefore the GDOP (geometric dilution of precision) is obtained as

$$
\begin{align*}
\operatorname{GDOP}(x, y) & =\sqrt{\operatorname{tr}(\mathbf{P})} \\
& =\sqrt{\left(T_{11}^{2}+T_{21}^{2}\right) B_{1}^{2} \sigma_{\theta 1}^{2}+\left(T_{12}^{2}+T_{22}^{2}\right) B_{2}^{2} \sigma_{\theta 2}^{2}} \tag{7.9}
\end{align*}
$$

Based on

$$
\mathbf{T}=\left[\begin{array}{ll}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{array}\right]=\frac{1}{\sin \left(\theta_{1}-\theta_{2}\right)}\left[\begin{array}{cc}
-\cos \theta_{1} \cos \theta_{2} & \cos \theta_{1} \cos \theta_{2} \\
\cos \theta_{1} \sin \theta_{2} & \sin \theta_{1} \cos \theta_{2}
\end{array}\right],
$$

by substituting it into the expression above we can obtain

$$
\begin{equation*}
G D O P(x, y)=\frac{1}{\left|\sin \left(\theta_{1}-\theta_{2}\right)\right|} \sqrt{r_{1}^{2} \sigma_{\theta 1}^{2}+r_{2}^{2} \sigma_{\theta 2}^{2}} \tag{7.10}
\end{equation*}
$$

where $r_{1}$ and $r_{2}$ are the distances from the emitter to the two observers. The above expression indicates that the error of triangulation is equal to the quadratic sum of the polygon's side lengths of the crossing area of LOSs divided by the sine of their included angle, as shown in Figure 7.3.
The circular error probability (CEP) is typically used as an indicator, which has an approximately relationship with the GDOP as

$$
\begin{equation*}
\operatorname{CEP}(x, y)=0.75 G D O P(x, y)=\frac{0.75}{\left|\sin \left(\theta_{1}-\theta_{2}\right)\right|} \sqrt{r_{1}^{2} \sigma_{\theta 1}^{2}+r_{2}^{2} \sigma_{\theta 2}^{2}} \tag{7.11}
\end{equation*}
$$



Figure 7.3 Area of triangulation error

When the distance between the two stations is 40 km , the error distribution of triangulation is as shown in Figure 7.4.
Figure 7.4 shows that the geolocation error in the line direction of the two stations approaches infinity, so it cannot be located in the line direction between the two stations. However, the best geolocation accuracy is perpendicular to the direction of the line between the two stations. Note that the contour value in Figure 7.4a is flatter than that in Figure 7.4b, obviously showing that the distributions for both of them are slightly different.

### 7.2.3 Optimal Geometric Configuration of Observers

A series of simplified assumptions may be made to find the optimal geometric configuration of observers for 2D triangulation. For simplicity, it is assumed that the two observer stations have the same angle measuring accuracy, that is, $\sigma_{\theta}=\sigma_{\theta 1}=\sigma_{\theta 2}$. The origin of the reference coordinate system is at the center of the line between the two stations, where the $x$ axis represents the direction of the line between the two stations and the $y$ axis conforms to the right-hand rule. As a result, the positions of the two stations are symmetrical in the Cartesian coordinate system, that is, $(-l, 0)$ and $(l, 0)$. Now the question becomes one of finding a point that has a minimum error for triangulation when the emitter is at this point.

Therefore, expression (7.10) can be further simplified as

$$
\begin{equation*}
\operatorname{GDOP}(x, y)=\frac{\sigma_{\theta}}{\left|\sin \left(\theta_{1}-\theta_{2}\right)\right|} \sqrt{r_{1}^{2}+r_{2}^{2}} \tag{7.12}
\end{equation*}
$$

Based on law of sines, we find that

$$
\left\{\begin{array}{l}
r_{1}=\frac{2 l \sin \theta_{2}}{\sin \left(\theta_{1}-\theta_{2}\right)}  \tag{7.13}\\
r_{2}=\frac{2 l \sin \theta_{1}}{\sin \left(\theta_{1}-\theta_{2}\right)}
\end{array}\right.
$$



Figure 7.4 Error distribution of triangulation, $\sigma_{\theta}=1^{\circ}$. (a) Distribution of the absolute geolocation error (CEP, km) and (b) distribution of the relative geolocation error (CEP, $\% R$ )

After expression (7.13) is substituted into expression (7.12), we find that:

$$
\begin{equation*}
G D O P(x, y)=2 l \sigma_{\theta} \frac{\sqrt{\sin ^{2} \theta_{1}+\sin ^{2} \theta_{2}}}{\sin ^{2}\left(\theta_{1}-\theta_{2}\right)} \tag{7.14}
\end{equation*}
$$

To find the minimum values of $\theta_{1}$ and $\theta_{2}$, the following extremum must be satisfied:

$$
\left.\begin{array}{l}
\frac{\partial G D O P}{\partial \theta_{1}}=0  \tag{7.15}\\
\frac{\partial G D O P}{\partial \theta_{2}}=0
\end{array}\right\}
$$

Then the equation may be established:

$$
\left.\begin{array}{r}
\sin \theta_{1} \cos \theta_{1} \sin \left(\theta_{1}-\theta_{2}\right)=2\left(\sin ^{2} \theta_{1}+\sin ^{2} \theta_{2}\right) \cos \left(\theta_{1}-\theta_{2}\right)  \tag{7.16}\\
\sin \theta_{2} \cos \theta_{2} \sin \left(\theta_{1}-\theta_{2}\right)=-2\left(\sin ^{2} \theta_{1}+\sin ^{2} \theta_{2}\right) \cos \left(\theta_{1}-\theta_{2}\right)
\end{array}\right\} .
$$

We find that

$$
\begin{equation*}
\sin 2 \theta_{1}=\sin \left(-2 \theta_{2}\right) \tag{7.17}
\end{equation*}
$$

If only the range of $0 \leq \theta_{2} \leq \pi / 2$ is considered, there are three conditions that can be discussed:

1. If $\theta_{1}=-\theta_{2}$, the two lines do not intersect and the extreme value of the localization error at this point is the maximum value.
2. If $\theta_{1}=\theta_{2}=0$ or $\theta_{1}=\theta_{2}=\pi / 2$, the two DF lines are parallel but do not intersect, so the value is also the maximum value.
3. If $\theta_{1}=\pi-\theta_{2}$, that is, the two DF lines intersect as an isosceles triangle, the point of intersection is at the $y$ axis, that is, the optimal geolocation accuracy is at $x=0$. Make $\theta_{1}=\pi-\theta_{2}=\theta$ the lowest geolocation error that can be obtained by substituting expression (7.14):

$$
\begin{equation*}
G D O P(x, y)=\frac{\sqrt{2} l \sigma_{\theta}}{\left|2 \sin \theta \cos ^{2} \theta\right|} . \tag{7.18}
\end{equation*}
$$

The GDOP takes the minimum value when $\sin \theta \cos ^{2} \theta$ has the maximum value. By differentiating $\sin \theta \cos ^{2} \theta$ with respect to $\theta$ and letting the derivative equal 0 , we can obtain

$$
\begin{equation*}
\theta_{G D O P \max }=\operatorname{arctg}(1 / \sqrt{2})=35.3^{\circ} \tag{7.19}
\end{equation*}
$$

Conclusion 1 can be drawn from the above:

Conclusion 1. If the distance between two DF stations is specified, when the emitter is at the central line of two DF stations and their included angle is about $109.4^{\circ}$, the error for triangulation reaches the minimum value. After expression (7.14) is transformed based on the trigonometric function, another expression for the GDOP can be obtained:

$$
\begin{equation*}
G D O P(x, y)=\frac{R \sigma_{\theta}}{\sin \left(\theta_{1}-\theta_{2}\right)} \sqrt{\frac{1}{\sin ^{2} \theta_{1}}+\frac{1}{\sin ^{2} \theta_{2}}} \tag{7.20}
\end{equation*}
$$

where $R$ represents the distance from the emitter to the center of two DF stations. The lowest geolocation error can be obtained by repeating the same process as Conclusion 1:

$$
\begin{equation*}
\operatorname{GDOP}(x, y)=\frac{\sqrt{2} l \sigma_{\theta}}{\left.\mid 2 \sin ^{2} \theta \cos \theta\right) \mid} \tag{7.21}
\end{equation*}
$$

that is, the GDOP is the minimum value when $\sin ^{2} \theta \cos \theta$ obtains the maximum value. By differentiating $\sin ^{2} \theta \cos \theta$ with respect to $\theta$ and letting the derivative equal 0 , we can obtain

$$
\begin{equation*}
\theta_{G D O P \max }=\operatorname{arctg} \sqrt{2}=54.7^{\circ} \tag{7.22}
\end{equation*}
$$

Conclusion 2 is drawn from the above:
Conclusion 2. If the emitter distance $R$ is specified, when the emitter is at the central line of two DF stations and their included angle is about $70.6^{\circ}$, the error for triangulation reaches the minimum value.In many cases, the relative geolocation accuracy $(\% R)$ is used for measuring the geolocation accuracy of the passive geolocation system. By defining the center point of the line between the two stations as the origin, the relative geolocation error can be obtained as

$$
\begin{equation*}
\frac{G D O P(x, y)}{R}=\frac{\sigma_{\theta}}{\sin \left(\theta_{1}-\theta_{2}\right)} \sqrt{\frac{1}{\sin ^{2} \theta_{1}}+\frac{1}{\sin ^{2} \theta_{2}}} \tag{7.23}
\end{equation*}
$$

The analyzed result is the same as Conclusion 2, that is, the condition for the relative geolocation error to be the minimum value can be obtained by differentiating $\sin ^{2} \theta \cos \theta$ with respect to $\theta$ and letting the derivative equal 0 :

$$
\begin{equation*}
\theta_{G D O P \max }=\operatorname{arctg} \sqrt{2}=54.7^{\circ} \tag{7.24}
\end{equation*}
$$

Conclusion 3 is drawn from the above:
Conclusion 3. When the emitter is at the central line of two DF stations and their included angle is about $70.6^{\circ}$, the relative error for triangulation reaches the minimum value.


Figure 7.5 The optimal geolocation accuracy under different geolocation conditions. (a) The optimal locating point when the space between the reconnaissance stations is specified and (b) the optimal station deployment when the reconnaissance distance is specified

Owing to the fact that judging rules for optimal station deployment are different, the results of Conclusions 1 to 3 are not the same. As shown in Figure 7.5, when the reconnaissance distance is specified, the two stations should be preferably located at $\mathrm{A}_{1}$ and $\mathrm{A}_{2}$ to ensure that the emitter located at point T with a distance of $R$ has the minimum geolocation error. Although the minimum geolocation error of the triangulation system at $\mathrm{A}_{1}$ and $A_{2}$ is not obtained at point $T$ but at point $P$, the geolocation error at that time has been the minimum error for all modes of deployment under the equal distance. When the relative error index is used, the optimal relative geolocation accuracy of the triangulation system is obtained at point T .

### 7.3 Multiplatform TDOA Geolocation

### 7.3.1 Theory of Multiplatform TDOA Geolocation

TDOA geolocation, also known as hyperbola geolocation, is achieved by processing the TDOA data of the arrival time of the signals that are transmitted from the emitter and are received by multiplatforms. The hyperbola TDOA system consists of one main platform (station) and more than two auxiliary stations. The geolocation theory of the TDOA is shown in Figure 7.6 [5].
The geometric configuration of the passive TDOA system is shown as Figure 7.6a. This system consists of a main station and two or three auxiliary stations, where the main station is not only receiving signals directly from an emitter, but is receiving the transmitted signals forwarded from the auxiliary stations. These signals are processed in the main station, so the TDOA between the arrival time when the transmitted signal arrived at the main station and that of transmitted signal when it arrived at the auxiliary station is obtained. The location of the emitter can be estimated after intercepting and recognition of the emitter signal. The receiver between the auxiliary station and the main station receives signals transmitted from the emitter and forwards these signals to the main station. The baselines are the lines between the main station and the auxiliary stations. The angle of the baseline is the included angle between the baselines.


Figure 7.6 Theory of hyperbola TDOA geolocation. (a) Geometric theory of TDOA geolocation and (b) pulse signal waveform of each station

For a radar pulse signal, if the locations of the main station and each substation in Figure 7.6a are known and all stations receive transmitted signals from a radar emitter, correspondingly the TDOA between the arrival time that the signal transmitted from the same emitter arrives at the main station and that arrives at each station can be measured (Figure 7.6b), so the measured time difference from the main station C to the auxiliary station A is proportional to the distance difference from the emitter to the two stations. As a result, one hyperbola line $L_{1}$ focused at station A and station C can be determined, and the other hyperbola line $\mathrm{L}_{2}$ can also be determined by the TDOA measured between the main station C and the other auxiliary station B. The point of intersection is the place where the emitter is located. It can be seen that the hyperbola TDOA geolocation system consists of at least three receivers, in order to achieve the geolocation for an emitter on a 2D plane. For a long-distant moving emitter, the approximately 2D geolocation algorithm can be used, even if errors exist.
The TDOA geolocation technique features high accuracy and is independent of the carrier frequency of the signal, which is helpful when forming an accurate flight track. In addition, the system covers a certain area of a sector with high accuracy. Currently the typical geolocation accuracy of an existing system is specified as follows: for an emitter from 150 km right ahead, the distance error (RMS error value) is approximately 250 m and the tangential error (RMS (root mean square) error value) is approximately 25 m [5].

### 7.3.2 2D TDOA Geolocation Algorithm

When the time of arrival (TOA) of the same signal from an emitter and the two TDOAs are measured by at least three observers in a group, the emitter can be located in a 2 D plane, shown as Figure 7.7. If the geolocation system consists of one main station and $n(n \geq 2)$ auxiliary stations, the 2D geolocation for an emitter can be achieved. The location of each observer station is written as $\left(x_{j}, y_{j}\right)^{\mathrm{T}}, j=0,1,2, \ldots, n$, where $j=0$ represents the main station and $j=1,2, \ldots, n$ represents the auxiliary station. Suppose the location of the emitter is $(x, y)^{\mathrm{T}}$ and $r_{j}$ is the distance between the emitter and the $j$ th $(j=0,1,2, \ldots, n)$ station. Then $\Delta r_{i}$ is


Figure 7.7 Sketch of TDOA geolocation by three observers
the distance difference between the emitter to the $j$ th station with an expression as follows [6]:

$$
\left.\begin{array}{l}
r_{0}^{2}=\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}  \tag{7.25}\\
r_{j}^{2}=\left(x-x_{j}\right)^{2}+\left(y-y_{j}\right)^{2}(j=1,2, \ldots, n) \\
\Delta r_{j}=r_{j}-r_{0}
\end{array}\right\}
$$

The above expression can be simplified as

$$
\begin{equation*}
\left(x_{0}-x_{j}\right) x+\left(y_{0}-y_{j}\right) y=k_{j}+r_{0} \Delta r_{j} \tag{7.26}
\end{equation*}
$$

where

$$
\begin{equation*}
k_{j}=\frac{1}{2}\left[\Delta r_{j}^{2}+\left(x_{0}^{2}+y_{0}^{2}\right)-\left(x_{j}^{2}+y_{j}^{2}\right)\right](j=1,2, \ldots, n) . \tag{7.27}
\end{equation*}
$$

Expression (7.26) shows a nonlinear equation set constructed by $n$ equations, with a matrix form of

$$
\begin{equation*}
\mathbf{A X}=\mathbf{F} \tag{7.28}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathbf{A}=\left[\begin{array}{cc}
x_{0}-x_{1} & y_{0}-y_{1} \\
\vdots & \vdots \\
x_{0}-x_{n} & y_{0}-y_{n}
\end{array}\right],  \tag{7.29}\\
\mathbf{X}=[x, y]^{\mathrm{T}}  \tag{7.30}\\
\mathbf{F}=\left[\begin{array}{c}
k_{1}+r_{0} \Delta r_{1} \\
\vdots \\
k_{n}+r_{0} \Delta r_{n}
\end{array}\right] . \tag{7.31}
\end{gather*}
$$

When $n=2$, that is, for the three-station TDOA geolocation system, if $\operatorname{rank}(\mathbf{A})=2$, the location of emitter $\widehat{\mathbf{X}}$ is determined by

$$
\begin{equation*}
\widehat{\mathbf{X}}=\mathbf{A}^{-1} \mathbf{F} \tag{7.32}
\end{equation*}
$$

Due to errors measured by the observer position measurement and the TDOA measurement, $\widehat{\mathbf{X}}$ represents the estimated location of the emitter:

If $n>2$ and the station geometry is satisfied with $\operatorname{rank}(\mathbf{A})=2$, the location of the emitter $\widehat{\mathbf{X}}$ is obtained by solving expression (7.28) using the pseudo-inverse method:

$$
\begin{equation*}
\widehat{\mathbf{X}}=\left(\mathbf{A}^{\mathrm{T}} \mathbf{A}\right)^{-1} \mathbf{A}^{\mathrm{T}} \mathbf{F} . \tag{7.33}
\end{equation*}
$$

If $n=2$, let

$$
\mathbf{A}^{-1}=\left[\begin{array}{ll}
a_{11} & a_{12}  \tag{7.34}\\
a_{21} & a_{22}
\end{array}\right]=\left[a_{i j}\right]_{2 \times 2}
$$

If $n>2$, let

$$
\left(\mathbf{A}^{\mathrm{T}} \mathbf{A}\right)^{-1} \mathbf{A}^{\mathrm{T}}=\left[\begin{array}{lll}
a_{11} & \cdots & a_{1 n}  \tag{7.35}\\
a_{21} & \cdots & a_{2 n}
\end{array}\right]=\left[a_{i j}\right]_{2 \times n} .
$$

The following expression can be obtained based on Equations (7.32) and (7.33):

$$
\left.\begin{array}{l}
\hat{x}=m_{1}+n_{1} r_{0}  \tag{7.36}\\
\hat{y}=m_{2}+n_{2} r_{0}
\end{array}\right\}
$$

where

$$
\left.\begin{array}{ll}
m_{i}=\sum_{j=1}^{n} a_{i j} k_{j} &  \tag{7.37}\\
n_{i}=\sum_{j=1}^{n} a_{i j} \Delta r_{j} & (i=1,2)
\end{array}\right\}
$$

After expression (7.36) is substituted into the first equation in expression (7.25), we can obtain

$$
\begin{equation*}
\alpha r_{0}^{2}+2 \beta r_{0}+\gamma=0 \tag{7.38}
\end{equation*}
$$

where

$$
\left.\begin{array}{l}
\alpha=n_{1}^{2}+n_{2}^{2}-1  \tag{7.39}\\
\beta=\left(m_{1}-x_{0}\right) n_{1}+\left(m_{2}-y_{0}\right) n_{2} \\
\gamma=\left(m_{1}-x_{0}\right)^{2}+\left(m_{2}-y_{0}\right)^{2}
\end{array}\right\}
$$

Expression (7.38) is used to find the root of $r_{0}$ and then the solved $r_{0}$ is substituted into Equation (7.32) to measure the location of the emitter. In fact, two values $r_{01}$ and $r_{02}$ related to $r_{0}$ can be obtained through solution of Equation (7.38):

1. If $r_{01} r_{02}<0$, the positive one is taken as $r_{0}$.
2. If $r_{01}$ and $r_{02}$ are both positive values, the geolocation cannot be determined. It is called the ambiguous geolocation case.

Ambiguous geolocation may be caused by two points of intersection of a hyperbola crossing. In the case of three stations, some information is needed in order to ascertain the ambiguous point, such as an azimuth angle measured from one station. If $r_{01}$ and $r_{02}$ are substituted into expression (7.32) to obtain two points and then the values of two points are calculated to get two azimuth angles, by comparing the two azimuth angles with the measured angles, the value of $r_{0}$ is determined correctly.
Reference [6] gives an easier method for judging the ambiguous point. By using this method, only the location of the baseline against the emitter (in front or behind of the baseline) is required without other supporting measured data.
For the 2D geolocation system, the following conclusion is made based on the geometric algorithm. In a three-station geolocation system, two sets of hyperbolas determined by the geolocation equation have no more than two points of intersection. If there is only one intersected point, there is no ambiguous geolocation problem. However, if there are two points of intersection, the location of two points must be located on each side of the baseline. In a defense system, the TDOA system is generally deployed in front of the transmitter of the interested area, so that if an ambiguous geolocation exists during 2D geolocation of the transmitter, the locating point behind the baseline is not the true point.

### 7.3.3 TDOA Geolocation Using the Altitude Assumption

The 2D TDOA algorithm mentioned above is used only for geolocating the emitter placed in the same plane as observers. In the case of the three-station TDOA geolocation system, if only the 2D coordinates (i.e., $x$ and $y$ coordinates) of the emitter are required, the 3D geolocation of the emitter is achieved by assuming the height of the emitter via three observers. If the assumed altitude error of the emitter is not zero, there is a systematic error for the emitter geolocation, which is caused by the assumption above. We should make a reasonable altitude assumption based on prior knowledge or supporting data, in order to minimize the geolocation error caused by the assumption of the emitter altitude [6].
By assuming the emitter altitude, the three observers are utilized for 3D geolocating the emitter. The position of an observer is assumed to be $\left(x_{i}, y_{i}, z_{i}\right)^{\mathrm{T}}(i=0,1,2)$, where $i=0$ represents the main station and $i=1,2$ represents the auxiliary station. The position of the emitter is $(x, y, z)^{\mathrm{T}}$ and in 3D space is given as the expression

$$
\left.\begin{array}{c}
r_{0}^{2}=\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2}  \tag{7.40}\\
r_{i}^{2}=\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}+\left(z-z_{i}\right)^{2} \\
\Delta r_{i}=r_{i}-r_{0}
\end{array}\right\}(i=1,2)
$$

where $r_{i}$ is the distance between the emitter and the $i$ th station $(i=0,1,2)$ and $\Delta r_{i}$ is the difference in distance between the emitter and the $i$ th station. Expression (7.40) includes only two geolocation equations, which cannot be used to solve 3D geolocation problems. By properly assuming the altitude of the emitter, it is assumed that $z=\hat{z}$ for the emitter is known and can be substituted to solve the sets of equations (Equation (7.40)). Only a few steps should be modified, as in the algorithm in Section 7.3.2.
Expression (7.27) is modified to

$$
\begin{equation*}
k_{i}=\frac{1}{2}\left[\Delta r_{i}^{2}+\left(x_{0}^{2}+y_{0}^{2}\right)-\left(x_{i}^{2}+y_{i}^{2}\right)-\left(z_{0}-z_{i}\right) \hat{z}\right](i=1,2) . \tag{7.41}
\end{equation*}
$$

Expression (7.39) is modified to

$$
\left.\begin{array}{l}
\alpha=n_{1}^{2}+n_{2}^{2}-1  \tag{7.42}\\
\beta=\left(m_{1}-x_{0}\right) n_{1}+\left(m_{2}-y_{0}\right) n_{2} \\
\gamma=\left(m_{1}-x_{0}\right)^{2}+\left(m_{2}-y_{0}\right)^{2}+\left(\hat{z}-z_{0}\right)^{2}
\end{array}\right\}
$$

Then the geolocation of the emitter in 3D space using the altitude assumption can be realized.

### 7.3.4 3D TDOA Geolocation Algorithm

Only three stations are required for 2D geolocation of the emitter. If there is no assumption of the emitter altitude, 3D geolocation for the emitter needs at least four stations. Suppose the geolocation system consists of one main station and $n(n \geq 3)$ auxiliary stations and the position of the $j$ th station is $\left(x_{j}, y_{j}, z_{j}\right)^{\mathrm{T}}, j=0,1,2, \ldots, n$, where $j=0$ represents the main station and $j=1,2, \ldots, n$ represents the auxiliary station. The position of the emitter is $(x, y, z)^{\mathrm{T}}$, which is unknown, and $r_{j}$ is the distance between the emitter and the $j$ th $(j=0,1,2, \ldots, n)$
station. $\Delta r_{i}$ is the distance difference between the emitter to the $j$ th $(j=0,1,2 \ldots n)$ station and the emitter to the primary station, with the following expression:

$$
\left.\begin{array}{l}
r_{0}^{2}=\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}+\left(z-z_{0}\right)^{2}  \tag{7.43}\\
r_{i}^{2}=\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}+\left(z-z_{i}\right)^{2}(i=1,2, \ldots, n) \\
\Delta r_{i}=r_{i}-r_{0}
\end{array}\right\}
$$

Expression (7.43) can be simplified to

$$
\begin{equation*}
\left(x_{0}-x_{i}\right) x+\left(y_{0}-y_{i}\right) y+\left(z_{0}-z_{i}\right) z=k_{i}+r_{0} \Delta r_{i} \tag{7.44}
\end{equation*}
$$

where

$$
\begin{equation*}
k_{i}=\frac{1}{2}\left[\Delta r_{i}^{2}+\left(x_{0}^{2}+y_{0}^{2}+z_{0}^{2}\right)-\left(x_{i}^{2}+y_{i}^{2}+z_{i}^{2}\right)\right] \quad(i=1,2, \ldots, n) \tag{7.45}
\end{equation*}
$$

Expression (7.44) shows that nonlinear equations can be rewritten in matrix form as

$$
\begin{equation*}
\mathbf{A X}=\mathbf{F} \tag{7.46}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathbf{A}=\left[\begin{array}{ccc}
x_{0}-x_{1} & y_{0}-y_{1} & z_{0}-z_{1} \\
\vdots & \vdots & \vdots \\
x_{0}-x_{n} & y_{0}-y_{n} & z_{0}-z_{n}
\end{array}\right],  \tag{7.47}\\
\mathbf{X}=\left[\begin{array}{lll}
x & y & z
\end{array}\right]^{\mathrm{T}}  \tag{7.48}\\
\mathbf{F}=\left[\begin{array}{c}
k_{1}+r_{0} \Delta r_{1} \\
\vdots \\
k_{n}+r_{0} \Delta r_{n}
\end{array}\right] . \tag{7.49}
\end{gather*}
$$

When $n=3$, that is, for a four-station TDOA geolocation system, if $\operatorname{rank}(\mathbf{A})=3$, the location of the emitter is determined by

$$
\begin{equation*}
\mathbf{X}=\mathbf{A}^{-1} \mathbf{F} \tag{7.50}
\end{equation*}
$$

If $n>3$ and the station site is satisfied with $\operatorname{rank}(\mathbf{A})=3$, the following is obtained by solving expression (7.46) using the pseudo-inverse method:

$$
\begin{equation*}
\widehat{\mathbf{X}}=\left(\mathbf{A}^{\mathrm{T}} \mathbf{A}\right)^{-1} \mathbf{A}^{\mathrm{T}} \mathbf{F} \tag{7.51}
\end{equation*}
$$

If $n=3$, let

$$
\mathbf{A}^{-1}=\left[\begin{array}{lll}
a_{11} & a_{12} & a_{13}  \tag{7.52}\\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right]=\left[a_{i j}\right]_{3 \times 3}
$$

If $n>3$, let

$$
\left(\mathbf{A}^{\mathrm{T}} \mathbf{A}\right)^{-1} \mathbf{A}^{\mathrm{T}}=\left[\begin{array}{lll}
a_{11} & \cdots & a_{1 n}  \tag{7.53}\\
a_{21} & \cdots & a_{2 n} \\
a_{31} & \cdots & a_{3 n}
\end{array}\right]=\left[a_{i j}\right]_{3 \times n}
$$

Therefore, an estimation of the emitter can be computed based on Equations (7.50) and (7.51):

$$
\left.\begin{array}{l}
\hat{x}=m_{1}+n_{1} r_{0}  \tag{7.54}\\
\hat{y}=m_{2}+n_{2} r_{0} \\
\hat{z}=m_{3}+n_{3} r_{0}
\end{array}\right\},
$$

where

$$
\left.\begin{array}{l}
m_{i}=\sum_{j=1}^{n} a_{i j} k_{j}  \tag{7.55}\\
n_{i}=\sum_{j=1}^{n} a_{i j} \Delta r_{j}
\end{array}\right\}(i=1,2,3)
$$

After expression (7.54) is substituted into the first equation of expression (7.40), we obtain

$$
\begin{equation*}
\alpha r_{0}^{2}+2 \beta r_{0}+\gamma=0 \tag{7.56}
\end{equation*}
$$

where

$$
\left.\begin{array}{l}
\alpha=n_{1}^{2}+n_{2}^{2}+n_{3}^{2}-1  \tag{7.57}\\
\beta=\left(m_{1}-x_{0}\right) n_{1}+\left(m_{2}-y_{0}\right) n_{2}+\left(m_{3}-z_{0}\right) n_{3} \\
\gamma=\left(m_{1}-x_{0}\right)^{2}+\left(m_{2}-y_{0}\right)^{2}+\left(m_{3}-z_{0}\right)^{2}
\end{array}\right\} .
$$

Note that $r_{0}$ is solved by expression (7.56), and then the solved $r_{0}$ is substituted into Equation (7.50) or (7.5 to get the estimated value of the emitter location. Two values of $r_{01}$ and $r_{02}$ related to $r_{0}$ can be obtained through solution of the quadratic equation (Equation (7.56)), so the ambiguous geolocation also exists in 3D geolocation for the emitter. For the 3D geolocation system, the following conclusion can be made based on geometric theory. In a four-station TDOA geolocation system, two hyperbolas determined by the geolocation equation have no more than two points of intersection. If there is one intersected point, there is no ambiguous geolocation case. However, if there are two points of intersection, the location of the two points must be located on each side of the area constructed by the four stations.
When $n>3$, in addition to the methods above that eliminate the problem of ambiguous geolocation, there is another method to use: the TDOA measurement is divided into two subsets, the 3D geolocation of the emitter is performed, and then the minimum range matching criterion is used to recognize the false locating point. In fact, from the 3D station geometric configuration, $\operatorname{rank}(\mathbf{A})=3$ is required and the necessary condition of 3D geolocation demands that all observer stations should not deploy on the same plane.

### 7.4 Localization Theory by a Single Platform

Based on kinematic principles, there is relative movement between the emitter and the observer platform (a near-space platform such as a hypersonic vehicle), which has an antenna array with at least two spatial separated units (consisting of an interferometer) and can be utilized to measure the phase rate of changing (PRC) information of a transmitted electromagnetic wave from the unknown emitter with an unknown position. The intercepted signal contains the information that affects the location of the emitter. Then the azimuth angle and the elevation angle of the emitter measured by the attitude sensor are utilized for localization of the emitter [7].

### 7.4.1 Measurement Model of Localization

In a 3D Cartersian coordinates system, it is assumed that the vector of the baseline of the interferometer on the observer platform is

$$
\mathbf{d}=\left[\begin{array}{lll}
d_{x} & d_{y} & d_{z} \tag{7.58}
\end{array}\right]^{\mathrm{T}}
$$

where the length of the baseline of the interferometer is $d=\|\mathbf{d}\|$. The relative position vector between the observer and the emitter is

$$
\begin{align*}
\mathbf{X} & =\left[\begin{array}{ll}
x & y
\end{array}\right]^{\mathrm{T}} \\
& =\mathbf{X}_{T}-\mathbf{X}_{O} \tag{7.59}
\end{align*}
$$

where $\mathbf{X}_{T}=\left[\begin{array}{lll}x_{T} & y_{T} & z_{T}\end{array}\right]^{\mathrm{T}}$ and $\mathbf{X}_{O}=\left[\begin{array}{lll}x_{O} & y_{O} & z_{O}\end{array}\right]^{\mathrm{T}}$, and the cosine of the included angle between $\mathbf{X}$ and $\mathbf{d}$ is

$$
\begin{equation*}
\cos \theta_{A}=\frac{(\mathbf{d} \cdot \mathbf{X})}{\|\mathbf{d}\|\|\mathbf{X}\|}=\frac{\mathbf{d}^{\mathrm{T}} \mathbf{X}}{d\|\mathbf{X}\|}=\frac{d_{x} x+d_{y} y+d_{z} z}{d \sqrt{x^{2}+y^{2}+z^{2}}} \tag{7.60}
\end{equation*}
$$

From the definition of the phase difference of the interferometer in Equation (3.10), its phase difference equals the length of the propagation path divided by the wavelength $\lambda$ multiplied by $2 \pi$, that is

$$
\begin{align*}
\phi & =\bmod \left(2 \pi \frac{d}{\lambda} \cos \theta_{A}, 2 \pi\right) \\
& =\bmod \left(\frac{2 \pi}{\lambda} \frac{d_{x} x+d_{y} y+d_{z} z}{\sqrt{x^{2}+y^{2}+z^{2}}}, 2 \pi\right) . \tag{7.61}
\end{align*}
$$

Assume that an interferometer is mounted on a near-space platform, as shown in Figure 7.8. The heading (yaw) angle of the near-space platform is $\alpha$. The pitch angle is $\theta$. By assuming that the baseline has a fixed azimuth angle $\psi$ and a fixed elevated angle $\gamma$ between the axis of platform, the vector of the baseline can be measured as

$$
\left.\begin{array}{l}
d_{x}=d \sin (\alpha-\psi) \cos (\theta-\gamma)  \tag{7.62}\\
d_{y}=d \cos (\alpha-\psi) \cos (\theta-\gamma) \\
d_{z}=d \sin (\theta-\gamma)
\end{array}\right\}
$$

The phase difference can be obtained as

$$
\begin{equation*}
\phi=\bmod \left(\frac{2 \pi d}{\lambda} \frac{\sin (\alpha-\psi) \cos (\theta-\gamma) x+\cos (\alpha-\psi) \cos (\theta-\gamma) y+\sin (\theta-\gamma) z}{\sqrt{x^{2}+y^{2}+z^{2}}}, 2 \pi\right) \tag{7.63}
\end{equation*}
$$

By differentiating the phase difference in Equation (7.63) with respect to time, we obtain

$$
\begin{array}{r}
\dot{\phi}=\frac{2 \pi}{\lambda}\left\{\frac { 1 } { ( x ^ { 2 } + y ^ { 2 } + z ^ { 2 } ) ^ { 3 / 2 } } \left[d_{x} \dot{x}\left(y^{2}+z^{2}\right)+d_{y} \dot{y}\left(x^{2}+z^{2}\right)+d_{z} \dot{z}\left(x^{2}+y^{2}\right)-x \dot{x}\left(y d_{y}+z d_{z}\right)\right.\right. \\
\left.\left.-y \dot{y}\left(x d_{x}+z d_{z}\right)-z \dot{z}\left(x d_{x}+y d_{y}\right)\right]+\frac{\dot{d}_{x} x+\dot{d}_{y} y+\dot{d}_{z} z}{\left(x^{2}+y^{2}+z^{2}\right)^{1 / 2}}\right\} . \tag{7.64}
\end{array}
$$



Figure 7.8 Established angle of the baseline of the interferometer diagram

It is considered that the mounted angle of the baseline remains the same, so the vector of the baseline is differentiated by

$$
\left.\begin{array}{l}
\dot{d}_{x}=d[\dot{\alpha} \cos (\alpha-\psi) \cos (\theta-\gamma)-\dot{\theta} \sin (\alpha-\psi) \sin (\theta-\gamma)]  \tag{7.65}\\
\dot{d}_{y}=d[-\dot{\alpha} \sin (\alpha-\psi) \cos (\theta-\gamma)-\dot{\theta} \cos (\alpha-\psi) \sin (\theta-\gamma)] \\
\dot{d}_{z}=d \dot{\theta} \cos (\theta-\gamma)
\end{array}\right\}
$$

The vector is substituted to obtain a measurement equation for the PRC.
As shown in Figure 7.9, the angle of arrival (AOA) interferometer with a short baseline at that time can be measured as

$$
\begin{align*}
\theta_{A, m} & =\arccos \left(\frac{d_{x} x+d_{y} y+d_{z} z}{d \sqrt{x^{2}+y^{2}+z^{2}}}\right)+\delta_{\theta_{A}} \\
& =\arccos \left(\frac{\sin (\alpha-\psi) \cos (\theta-\gamma) x+\cos (\alpha-\psi) \cos (\theta-\gamma) y+\sin (\theta-\gamma) z}{\sqrt{x^{2}+y^{2}+z^{2}}}\right)+\delta_{\theta_{A}}, \tag{7.66}
\end{align*}
$$

where $\delta_{\theta_{A}}$ is the angle measurement error of the interferometer with a short-length baseline.

### 7.4.2 A 2D Approximate Localization Method

For 2D localization, compared by the distance of the emitter, the flying height of the aircraft is very low. It could approximately be considered as a 2D plane localization for the aircraft and


Figure 7.9 Direction finding of an interferometer on the 2D plane
the emitter. It is assumed that the altitude of the target emitter $z_{T}$ and the altitude of observation aircraft $z_{O}$ are known, so if $z=z_{T}-z_{O}$ is known, only $\mathbf{X}_{T}=\left(\begin{array}{ll}x_{T} & y_{T}\end{array}\right)^{\mathrm{T}}$ needs to be estimated. Under this assumption, the elevation angle and the elevated angle of the mounted interferometer could be assumed to be zero only when the interferometer is moving on the horizontal plane, that is, $\theta=\gamma=0$. The actual angle of the interferometer is measured as

$$
\begin{align*}
\theta_{A, m} & =\arccos \left(\frac{\sin \left(\alpha_{m}-\psi\right) x+\cos \left(\alpha_{m}-\psi\right) y}{\sqrt{x^{2}+y^{2}+z^{2}}}\right)+\delta_{\theta_{A}} \\
& =f_{\theta}(\mathbf{X})+\delta_{\theta} \tag{7.67}
\end{align*}
$$

where $\delta_{\theta_{A}}$ is a measured angle error of the interferometer and $\mathbf{X}=\left(\begin{array}{ll}x & y\end{array}\right)^{\mathrm{T}}=\mathbf{X}_{T}-\mathbf{X}_{O}$. If $z=0$, the angle of the interferometer is measured approximately as

$$
\begin{equation*}
\theta_{A, m}=\alpha-\psi-\operatorname{arctg} \frac{x}{y}+\delta_{\alpha}+\delta_{\theta_{A}} \tag{7.68}
\end{equation*}
$$

In fact, $z \neq 0$, so only the model of expression (7.67) is used for localization. The ambiguous phase difference can be modeled as

$$
\begin{equation*}
\phi_{m}=\frac{2 \pi d}{\lambda} \frac{\sin (\alpha-\psi) x+\cos (\alpha-\psi) y}{\sqrt{x^{2}+y^{2}+z^{2}}}-N 2 \pi+\delta_{\phi} \triangle f_{\phi}(\mathbf{X}, N)+\delta_{\phi} \tag{7.69}
\end{equation*}
$$

where $N$ is an unknown integer.
If the observer is moving horizontally, the PRC is

$$
\left.\begin{array}{c}
+\dot{\alpha} \frac{\cos (\alpha-\psi) x-\sin (\alpha-\psi) y}{\left(x^{2}+y^{2}+z^{2}\right)^{1 / 2}} \\
\triangle f_{\dot{\phi}}(\mathbf{X})+\delta_{\dot{\phi}} \tag{7.70}
\end{array}\right\}+\delta_{\dot{\phi}}
$$

The measurement model of localization for the angle and the PRC is constructed from the expressions (7.68) and (7.70).
For the localization process mentioned above, there is another problem in practical measurement, which is that the phase difference $\phi$ has a period of $2 \pi$ and if the phase difference exceeds $2 \pi$, ambiguity might happen. The minimum length of the baseline $d$ is usually used for an unambiguous wide angle of sight. The maximum distance between the two antennas without ambiguity is $d_{\max }=\lambda / 2$.
For an interferometer used in single-platform high accuracy localization, to obtain a larger baseline-to-wavelength ratio, the distance of baseline $d$ is typically far longer than the maximum distance of the unambiguous baseline length $d_{\text {max }}$, so the phase measured may contain ambiguity. We usually combine several interferometers with different lengths to solve the ambiguity problem. However, we can select another method. As only the PRC is required for localization, apparently the PRC is derived from the received original phase difference (differential), but an unknown number of $2 \pi$ could appear because of the ambiguous phase. As a result, the phase difference should be solved for its ambiguity. If the baseline of the interferometer is long, the ambiguity cannot be solved by a single-base-line interferometer. If the same signal is received multiple times, the ambiguous phase differences measured at different times would be utilized to solve the PRC without time-domain ambiguity.
As shown in Figure 7.10, the measured phase difference $\phi$ with a range of $[0,2 \pi)$ shows ambiguity. The difference between the two phases can be solved at adjacent times if a less than $\pi$ jump appears. The unambiguous phase difference rate of the changing result could then be obtained. Therefore, if the rate of changing between the two phases does not exceed $\pi$, there will be no ambiguity in the PRC measurement. If exceeded, the phase compensation $\pm 2 \pi$ is needed to obtain the correct PRC for solving ambiguity.

### 7.4.3 MGEKF (Modified Gain Extended Kalman Filter) Localization Method

Based on the models above, a nonlinear filtering method such as the EKF (extended Kalman filter) in Section 3.3.1 can be used to estimate the emitter location.
From the previous section, we obtained the 3D measurement model for an approximate angle and the PRC. Based on the models above, the effect of the observer altitude $z$ should be considered. In practice, normally the distance of the emitter is far more than the altitude of observer; thus the localization model may be seen as a 2 D localization condition.
For simplicity, the state equation and measurement equation can be modified. Let the observer state be defined as $\mathbf{X}_{O i}=\left[\begin{array}{llll}x_{O i} & y_{O i} & \dot{x}_{O i} & \dot{y}_{O i}\end{array}\right]^{\mathrm{T}}$ and the emitter state vector as


Figure 7.10 Time-domain ambiguity solved by the phase rate of changing
$\mathbf{X}_{T i}=\left[\begin{array}{llll}x_{T i} & y_{T i} & \dot{x}_{T i} & \dot{y}_{T i}\end{array}\right]^{\mathrm{T}}$. Therefore the state equation may modified as

$$
\begin{equation*}
\mathbf{X}_{T i}=\boldsymbol{\Phi}_{i, i-1} \mathbf{X}_{T i-1} \tag{7.71}
\end{equation*}
$$

The state transition matrix is

$$
\boldsymbol{\Phi}_{i, i-1}=\left[\begin{array}{cc}
\mathbf{I}_{2} & \left(t_{i}-t_{i-1}\right) \mathbf{I}_{2} \\
\mathbf{O}_{2} & \mathbf{I}_{2}
\end{array}\right],
$$

where $\mathbf{I}_{2}$ and $\mathbf{O}_{2}$ are the $2 \times 2$ identity matrix and the $2 \times 2$ zero matrix, respectively.


Figure 7.11 Scenario of localization simulation

The observed results in expressions (7.68) and (7.70) can be used as observation models, in such a way that if the observed condition is satisfied, the estimated values of $\widehat{\mathbf{X}}_{T 0}$ and $\mathbf{P}_{0}$ can first be measured by the first two measurements and then the estimated value of filter $\widehat{\mathbf{X}}_{T i}$ can be obtained by the EKF recursion formula in Section 3.3.1.

### 7.4.4 Simulation

The scenario of the single-platform passive localization simulation is shown in Figure 7.11.
To simplify the analysis, the following assumptions are made. The azimuth angle of the mounted interferometer is $0^{\circ}$ (in the same direction as the platform movement). The elevation angle of the mounted interferometer is $0^{\circ}$. The observer is moving with uniform linear motion. In the case of a fixed emitter, the PRC is calculated by the least squares method. The EKF method for the angle and the PRC in Section 7.4.3 is utilized to locate the emitter. By repeating the Monte Carlo tests to calculate the CEP 100 times, the statistical curve of the localization error versus time is calculated, as shown in Figure 7.11. It is assumed that the DF accuracy of the system is $1^{\circ}$. The measured accuracy of the phase difference is $10^{\circ} / \mathrm{s}$. The altitude of the operating near-space platform is 5 km , with speeds of $(250,0,0) \mathrm{km} / \mathrm{h}$ at the $x, y$ and $z$ axes, respectively. The random error of the platform attitude is $0.2^{\circ}$, with $0.003^{\circ} / \mathrm{s}$ of attitude rate. The random error of the platform location is 15 m and the platform speed is $0.5 \mathrm{~m} / \mathrm{s}$. The initial location of the observer is at $\left(118.8^{\circ} \mathrm{E}, 24.1^{\circ} \mathrm{N}\right)$ and the emitter is at $\left(119^{\circ} \mathrm{E}, 25^{\circ} \mathrm{N}\right)$. The data measurement rate is 1000 times per second and the signal frequency $f=2 \mathrm{GHz}$.
At first it is assumed that the length of baseline is $d=10 \mathrm{~m}$. The phase difference curve of the interferometer is shown in Figure 7.12 and the angle and the PRC curve are shown in Figure 7.13.


Figure 7.12 Phase difference measurement curve


Figure 7.13 Measurement curve of the angle and the phase rate of changing


Figure 7.14 Relative localization error ( $\% R$ ) curve by repeating the Monte Carlo tests 100 times

By repeating the Monte Carlo localization tests 100 times, the statistical curve of the relative localization error $(\% R)$ can be obtained, as shown in Figure 7.14.
In Figure 7.14, the dotted line shows a curve of the CRLB (Cramér-Rao lower bound) versus time and the solid line is a curve of the localization error versus time by statistics. It may therefore take about 40 seconds to converge below $2 \% R$ of relative range error. This shows that to obtain high accuracy of localization needs much more time for single-platform localization.

## References

1. Wang, Y., Li, D., and An, Y. (2009) Application of near-space vehicle in electronic reconnaissance. Aerospace Electronic Warfare, 4(25): P18-P24 (in Chinese).
2. Yi, Z. and Li, Q. (2006) Analysis of near-space vehicle and its military application. Journal of the Academy of Equipment Command and Technology, 5(17): P64-P69 (in Chinese).
3. Li, Y., Yao, W., Zheng, W., et al. (2006) Type and characteristics of near-space system. Satellite Application, 3(14): P1-P6 (in Chinese).
4. Poisel, R.A. (2012) Electronic Warfare Target Location Methods, 2nd edn, Artech House.
5. Zhou, Y., An, W., Guo, F., et al. (2009) Principles of Electronic Warfare. Beijing: Publishing House of Electronics Industry (in Chinese).
6. Yang, L. (1998) Research on Passive TDOA Geolocation and Its Signal Processing. Changsha: Graduate School of National University of Defense Technology (in Chinese).
7. Sun, Z., Guo, F., Feng, D., et al. (2008) Passive Localization and Tracking Technology by Single Observer. Beijing: National Defence Industry Press. (in Chinese).

## 8

## Satellite-to-Satellite Passive Orbit Determination by Bearings Only

### 8.1 Introduction

For surveillance of a space target such as a satellite or debris, angle information is the basic measurement that can be acquired in comparison with frequency information. For example, the angle information or line-of-sight (LOS), can be acquired using the direction finding method as described in Section 3.1 or optical imaging by the camera on the satellite. Therefore, passive tracking of a satellite target by the satellite using a bearings-only measurement is possible.
Assuming that there is one observing satellite and one target satellite, respectively (see Figure 8.1 for the relative geometrical relationship), the observing satellite measures the LOS of the target signal. Thus the orbit of the target satellite can be inferred through accumulating angle information of a certain period of time [1].
According to the definition of angle (LOS) measurement information, it is correlated with the position and the velocity component of the target satellite. Therefore, it is possible to determine the position and velocity estimation of the target satellite based on angle measurement information. To avoid the singularity issue possibly caused by adopting the classical orbit element as the state variable, as suggested in reference [1], in this chapter the position and velocity vector of the target satellite under the J2000.0 coordinate system are used as the state variables. Two models of the satellite-to-satellite bearings-only passive tracking problem are established under two system models: one is for the two-body model and the other takes the $J_{2}$ perturbation into consideration (i.e., assuming that the earth is oblate). The corresponding tracking methods are derived in detail.

### 8.2 Model and Method of Bearings-Only Passive Tracking

In this section, the position and velocity vector of the target satellite in the J2000.0 coordinate system are used as the state variable. Firstly, the state model and measurement model of satellite-to-satellite passive bearings-only tracking under the simplest two-body model are


Figure 8.1 Schematic sketch of satellite-to-satellite passive orbit determination
analyzed. This is followed by the situation of $J_{2}$ perturbation assuming that the earth is oblate, which leads to asymmetry of the earth's gravitational field. Then the EKF (extended Kalman filter) tracking method is also derived in detail.

### 8.2.1 Mathematic Model in the Case of the Two-Body Problem

### 8.2.1.1 State Model

Take the position and velocity vector of the target satellite in the inertial system J2000.0 as the state variable $\boldsymbol{X}$, that is,

$$
\boldsymbol{X}=\left[\begin{array}{ll}
\boldsymbol{r}^{\mathrm{T}} & \dot{\boldsymbol{r}}^{\boldsymbol{T}} \tag{8.1}
\end{array}\right]^{\mathrm{T}}
$$

Therefore, the state equation of satellite motion is expressed as

$$
\dot{\boldsymbol{X}}=\boldsymbol{F}(\boldsymbol{X})=\left[\begin{array}{ll}
\dot{\boldsymbol{r}}^{\mathrm{T}} & \ddot{\boldsymbol{r}}^{\mathrm{T}} \tag{8.2}
\end{array}\right]^{\mathrm{T}},
$$

where $\boldsymbol{r}=\left[\begin{array}{lll}x & y & z\end{array}\right]^{\mathrm{T}}$ and $\dot{\boldsymbol{r}}=\left[\begin{array}{lll}\dot{x} & \dot{y} & \dot{z}\end{array}\right]^{\mathrm{T}}$ are the position and velocity vector of the target satellite in the J2000.0 coordinate system, respectively, and $\boldsymbol{F}(\cdot)$ is the nonlinear transformation of the state variable $\boldsymbol{X}$. In the two-body motion model, from expression (2.10) the satellite motion equation can be expressed as

$$
\begin{equation*}
\ddot{\boldsymbol{r}}=-\mu \frac{\boldsymbol{r}}{r^{3}}, \tag{8.3}
\end{equation*}
$$

where $\mu$ is the gravitation constant, also referred to as the Kepler constant, and $r=\sqrt{x^{2}+y^{2}+z^{2}}$ is the distance from the target satellite to the center of the earth.

### 8.2.1.2 Measurement Model

Assuming that the observing satellite could measure the azimuth angle $\beta_{k}$ and the elevation angle $\varepsilon_{k}$ of the target satellite at the $k$ th moment by means of optics or radio, their definitions
can be written as follows:

$$
\begin{align*}
& \beta_{k}=\arctan \left(\frac{\rho_{y}(k)}{\rho_{x}(k)}\right)+n_{\beta k},  \tag{8.4}\\
& \varepsilon_{k}=\arctan \left(\frac{\rho_{z}(k)}{\sqrt{\rho_{x}^{2}(k)+\rho_{y}^{2}(k)}}\right)+n_{\varepsilon k}, \tag{8.5}
\end{align*}
$$

where $n_{\beta}$ and $n_{\varepsilon}$ are measurement noises at the azimuth angle and the elevation angle, where the measurement noise is supposed to be zero mean and the variances are Gaussian white noises of $\sigma_{\beta}^{2}$ and $\sigma_{\varepsilon}^{2}$, respectively. Then $\rho=\left[\rho_{x}(k) \rho_{y}(k) \rho_{z}(k)\right]^{\mathrm{T}}$ is the relative position vector of the target satellite in the coordinate system of the observing satellite as the origin and $k=0$, $1, \ldots, N_{s}-1$ (where $N_{s}$ is the total number of points being observed).
The measurement vector is defined as $\boldsymbol{Z}(k)=\left[\beta_{k} \varepsilon_{k}\right]^{T}$, and the measurement vector could be expressed as a nonlinear function of state variable $\boldsymbol{X}$ :

$$
\begin{equation*}
\boldsymbol{Z}(k)=\boldsymbol{H}(\boldsymbol{X}(k))+\boldsymbol{n}(k) \tag{8.6}
\end{equation*}
$$

where $\boldsymbol{H}(\cdot)$ is the nonlinear transformation of the measurement vector consisting of expressions (8.4) and (8.5) about the state variable $\boldsymbol{X}$,

$$
\boldsymbol{n}(k)=\left[n_{\beta}(k) n_{\varepsilon}(k)\right]^{\mathrm{T}}
$$

is the measured noise vector of the azimuth angle and elevation angle, and its corresponding covariance matrix is expressed as

$$
E\left[\boldsymbol{n}(k) \boldsymbol{n}(k)^{\boldsymbol{T}}\right]=\boldsymbol{R}(k)=\left[\begin{array}{cc}
\sigma_{\beta}^{2} & 0 \\
0 & \sigma_{\varepsilon}^{2}
\end{array}\right] .
$$

### 8.2.2 Tracking Method in the Case of the Two-Body Model

As the state equation is a continuous nonlinear equation and the measurement equation is a discrete nonlinear equation, here the EKF can be adopted to estimate the state of the target satellite. Therefore, discretization and linearization are required for the state equation and the measurement equation.
After discretization of the state equation of the target satellite (Equation (8.2)), the following can be obtained:

$$
\begin{equation*}
\boldsymbol{X}(k+1)-\boldsymbol{X}(k)=\int_{t_{k}}^{t_{k+1}} \boldsymbol{F}(\boldsymbol{X}(t)) \mathrm{d} t \tag{8.7}
\end{equation*}
$$

When the time interval $t_{k+1}-t_{k}=T$ is short enough, $\boldsymbol{F}(\boldsymbol{X}(t))$ can be expanded to the Taylor series with time $t_{k}$ :

$$
\begin{equation*}
\boldsymbol{F}(\boldsymbol{X}(t)) \approx \boldsymbol{F}(\boldsymbol{X}(k))+\boldsymbol{A}(\boldsymbol{X}(k)) \cdot \boldsymbol{F}(\boldsymbol{X}(k))\left(t-t_{k}\right) \tag{8.8}
\end{equation*}
$$

where

$$
\boldsymbol{A}(\boldsymbol{X}(k))=\left.\frac{\partial \boldsymbol{F}(\boldsymbol{X}(t))}{\partial \boldsymbol{X}}\right|_{t=t_{k}}
$$

is a $6 \times 6$ matrix. Substitute expression (8.8) into expression (8.7) and the following can be obtained:

$$
\begin{equation*}
\boldsymbol{X}(k+1)=\boldsymbol{X}(k)+\boldsymbol{F}(\boldsymbol{X}(k)) T+\boldsymbol{A}(\boldsymbol{X}(k)) \cdot \boldsymbol{F}(\boldsymbol{X}(k)) \frac{T^{2}}{2}+\boldsymbol{W}(k), \tag{8.9}
\end{equation*}
$$

where $\boldsymbol{W}(k)$ is the state noise when applying linearization to the nonlinear state equation. This is on the same order with $|\boldsymbol{X}(k)-\widehat{\boldsymbol{X}}(k / k)|^{2}$, but we can approximately assume $\mathrm{E}[\boldsymbol{W}(k)]=0$ and suppose that $\mathrm{E}\left[\boldsymbol{W}(k) \boldsymbol{W}(k)^{\mathrm{T}}\right]=\boldsymbol{Q}$ is the state noise covariance matrix.
By expression (8.2) and vector differentiation law, $\boldsymbol{A}(\boldsymbol{X}(k)$ ) can be expressed as follows:

$$
\left.\begin{array}{rl}
\boldsymbol{A}(\boldsymbol{X}(k)) & =\left[\begin{array}{ccccccc}
\frac{\partial \dot{\boldsymbol{r}}}{\partial \boldsymbol{r}_{3 \times 3}} & \frac{\partial \dot{\boldsymbol{r}}}{\partial \dot{\boldsymbol{r}}_{3 \times 3}} \\
\frac{\partial \dot{\boldsymbol{r}}}{\partial \boldsymbol{r}_{3 \times 3}} & \frac{\partial \dot{\boldsymbol{r}}}{\partial \dot{\boldsymbol{r}}_{3 \times 3}}
\end{array}\right] \\
& =\left[\begin{array}{ccccccc}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\frac{\mu}{r^{3}}\left(\frac{3 x^{2}}{r^{2}}-1\right) & \frac{\mu}{r^{3}}\left(\frac{3 x y}{r^{2}}\right) & \frac{\mu}{r^{3}}\left(\frac{3 x z}{r^{2}}\right) & 0 & 0 & 0 \\
\frac{\mu}{r^{3}}\left(\frac{3 y x}{r^{2}}\right) & \frac{\mu}{r^{3}}\left(\frac{3 y^{2}}{r^{2}}-1\right) & \frac{\mu}{r^{3}}\left(\frac{3 y z}{r^{2}}\right) & 0 & 0 & 0
\end{array}\right]_{t=t_{k}}  \tag{8.10}\\
\frac{\mu}{r^{3}}\left(\frac{3 z x}{r^{2}}\right) & \frac{\mu}{r^{3}}\left(\frac{3 z y}{r^{2}}\right) \\
\frac{\mu}{r^{3}}\left(\frac{3 z^{2}}{r^{2}}-1\right) & 0
\end{array} 0_{0}\right]_{l},
$$

where $\boldsymbol{r}, \dot{\boldsymbol{r}}$, and $\ddot{\boldsymbol{r}}$ are respectively the position, velocity, and acceleration vector of the target satellite in the J2000.0 coordinate system.
Substitute the filtered state value $\widehat{\boldsymbol{X}}(k / k)$ into expressions (8) and (8.9), respectively. The state prediction equation can then be expressed as

$$
\begin{equation*}
\widehat{\boldsymbol{X}}(k+1 / k)=\widehat{\boldsymbol{X}}(k / k)+\boldsymbol{F}(\hat{\boldsymbol{X}}(k / k)) T+\boldsymbol{A}(\hat{\boldsymbol{X}}(k / k)) \cdot \boldsymbol{F}(\widehat{\boldsymbol{X}}(k / k)) \frac{T^{2}}{2} . \tag{8.11}
\end{equation*}
$$

According to the definition [2] of the state transition matrix $\boldsymbol{\Phi}\left(t, t_{k}\right)$, this can be expanded to the Taylor series with time $t_{k}$ :

$$
\begin{equation*}
\boldsymbol{\Phi}\left(t, t_{k}\right)=\boldsymbol{\Phi}\left(t_{k}, t_{k}\right)+\left.\frac{\mathrm{d} \boldsymbol{\Phi}\left(t, t_{k}\right)}{\mathrm{d} t}\right|_{t=t_{k}}\left(t-t_{k}\right)+O\left(t-t_{k}\right) \tag{8.12}
\end{equation*}
$$

where $O\left(t-t_{k}\right)$ means the higher order terms (HOT). Using the property of the state transition matrix, the following can be obtained [3]:

$$
\begin{gather*}
\boldsymbol{\Phi}\left(t_{k}, t_{k}\right)=\boldsymbol{I}  \tag{8.13}\\
\left.\frac{\mathrm{d} \boldsymbol{\Phi}\left(t, t_{k}\right)}{\mathrm{d} t}\right|_{t=t_{k}}=\boldsymbol{A}(\boldsymbol{X}(k)) \tag{8.14}
\end{gather*}
$$

Substitute the above two expressions into expression (8.12), respectively, which can be expressed as

$$
\begin{equation*}
\boldsymbol{\Phi}\left(t, t_{k}\right)=\boldsymbol{I}+\boldsymbol{A}(\boldsymbol{X}(k))\left(t-t_{k}\right)+O(\Delta t) . \tag{8.15}
\end{equation*}
$$

Likewise, after discretization of the continuous state transition matrix, the following can be obtained:

$$
\begin{equation*}
\boldsymbol{\Phi}(k+1 / k)=\boldsymbol{I}+\boldsymbol{A}(\hat{\boldsymbol{X}}(k / k)) T \tag{8.16}
\end{equation*}
$$

As the measurement equation is a discrete nonlinear equation, only linearization is required. The measurement equation can be expanded as a Taylor series at $\widehat{\boldsymbol{X}}(k+1 / k)$. As the angle is measured in the observing satellite body coordinate system and the state variable is in the J2000.0 coordinate system, the observing satellite body coordinate system needs to be converted to the $\mathbf{J} 2000.0$ coordinate system in order to calculate the Jacobian matrix.
Assume that $\boldsymbol{r}_{a x .}=\left[\begin{array}{lll}x_{a x .} & y_{a x .} & z_{a x .}\end{array}\right]^{\mathrm{T}}$ and $\dot{\boldsymbol{r}}_{a x .}=\left[\begin{array}{lll}\dot{x}_{a x .} & \dot{y}_{a x .} & \dot{z}_{a x x}\end{array}\right]^{\mathrm{T}}$ are respectively the position and velocity vector of the observing satellite (let the subscript ' $a x$ ' appear as $O$ ) and the target satellite (let the subscript ' $a x$ ' appear as $T$ ) in the J2000.0 earth centered inertial (ECI) coordinate system, while $\rho=\left[\begin{array}{lll}\rho_{x} & \rho_{y} & \rho_{z}\end{array}\right]^{T}$ and $\dot{\rho}=\left[\begin{array}{lll}\dot{\rho}_{x} & \dot{\rho}_{y} & \dot{\rho}_{z}\end{array}\right]^{T}$ are respectively the position and velocity vector of the target satellite in the observing satellite body coordinate system. From the above definition of the coordinate system, the expression for converting from the inertial system of the geocentric epoch J2000.0 to the coordinate system of the observing satellite centroid measurement station can be expressed as

$$
\begin{equation*}
\rho=\boldsymbol{G}^{\mathrm{T}}\left(\boldsymbol{r}_{T}-\boldsymbol{r}_{O}\right) \tag{8.17}
\end{equation*}
$$

where $\boldsymbol{G}$ is the transfer matrix between the coordinate systems. According to the definition of the satellite body coordinate system, the following can be obtained:

$$
\begin{align*}
\boldsymbol{G}(i, 3) & =-\boldsymbol{r}_{O} /\left|\boldsymbol{r}_{O}\right|,  \tag{8.18}\\
\boldsymbol{G}(j, 2) & =-\boldsymbol{r}_{O} \times \dot{\boldsymbol{r}}_{O} /\left|\boldsymbol{r}_{O} \times \dot{\boldsymbol{r}}_{O}\right|,  \tag{8.19}\\
\boldsymbol{G}(k, 1) & =\boldsymbol{G}(j, 2) \times \boldsymbol{G}(i, 3), \tag{8.20}
\end{align*}
$$

where $\boldsymbol{G}(i, 3), \boldsymbol{G}(j, 2)$, and $\boldsymbol{G}(k, 1)$ are columns 3,2 , and 1 of matrix $\boldsymbol{G}$, respectively.
By the vector differentiation law, the measured Jacobian matrix is expressed as

$$
\boldsymbol{H}(k+1 / k)=\left[\begin{array}{cc}
\frac{\partial \beta_{k}}{\partial \boldsymbol{r}_{1 \times 3}} & \frac{\partial \beta_{k}}{\partial \dot{\boldsymbol{r}}_{1 \times 3}}  \tag{8.21}\\
\frac{\partial \varepsilon_{k}}{\partial \boldsymbol{r}_{1 \times 3}} & \frac{\partial \varepsilon_{k}}{\partial \dot{\boldsymbol{r}}_{1 \times 3}}
\end{array}\right]=\left[\begin{array}{lll}
\frac{\partial \beta_{k}}{\partial \rho} \cdot \frac{\partial \rho}{\partial \boldsymbol{r}_{1 \times 3}} & 0_{1 \times 3} \\
\frac{\partial \varepsilon_{k}}{\partial \boldsymbol{\rho}} \cdot \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{r}_{1 \times 3}} & 0_{1 \times 3}
\end{array}\right]_{2 \times 6}
$$

By expressions (8.4), (8.5), and (8.17), the following can be obtained:

$$
\begin{align*}
\frac{\partial \beta_{k}}{\partial \rho} & =\left[\frac{-\rho_{y}}{\rho_{x}^{2}+\rho_{y}^{2}} \frac{\rho_{x}}{\rho_{x}^{2}+\rho_{y}^{2}} 0\right]_{1 \times 3}  \tag{8.22}\\
\frac{\partial \varepsilon_{k}}{\partial \rho} & =\left[\frac{-\rho_{x} \rho_{z}}{\rho_{r}^{2} \sqrt{\rho_{x}^{2}+\rho_{y}^{2}}} \frac{-\rho_{y} \rho_{z}}{\rho_{r}^{2} \sqrt{\rho_{x}^{2}+\rho_{y}^{2}}} \frac{\sqrt{\rho_{x}^{2}+\rho_{y}^{2}}}{\rho_{r}^{2}}\right]_{1 \times 3}  \tag{8.23}\\
\frac{\partial \rho}{\partial \boldsymbol{r}} & =\boldsymbol{G}_{3 \times 3}^{\mathrm{T}} \tag{8.24}
\end{align*}
$$

See expression (8.17) for the definitions of $\boldsymbol{\rho}$ and $\boldsymbol{G}, \rho_{r}=\sqrt{\rho_{x}^{2}+\rho_{y}^{2}+\rho_{z}^{2}}$ is the distance between the observing satellite and the target satellite, and $0_{1 \times 3}$ is the zero vector of $1 \times 3$.
After the state equation and measurement equation are linearized and discretized, they can be substituted into the EKF expression for iterative calculation [2]. For a simple expression, here the recursive moments are expressed as subscripts of the matrix:

1. Calculate the state prediction estimation $\widehat{\boldsymbol{X}}_{k+1 / k}$ for one step as given by expression (8.11) and calculate its corresponding covariance matrix by the following expression:

$$
\begin{equation*}
\boldsymbol{P}_{k+1 / k}=\boldsymbol{\Phi}_{k+1 / k} \boldsymbol{P}_{k / k} \boldsymbol{\Phi}_{k+1 / k}^{\mathbf{T}}+\boldsymbol{Q} \tag{8.25}
\end{equation*}
$$

Here matrix $\boldsymbol{Q}$ is the covariance of the error introduced when the nonlinear state equation is linearized. It can generally be selected as a smaller constant matrix [2] by experience.
2. Compute the gain matrix:

$$
\begin{equation*}
\boldsymbol{K}_{k+1}=\boldsymbol{P}_{k+1 / k} \boldsymbol{H}_{k+1 / k}^{\mathbf{T}}\left(\boldsymbol{H}_{k+1 / k} \boldsymbol{P}_{k / k} \boldsymbol{H}_{k+1 / k}^{\mathbf{T}}+\boldsymbol{R}_{k+1}\right)^{-1} \tag{8.26}
\end{equation*}
$$

3. Update the state and the corresponding covariance matrices:

$$
\begin{align*}
\hat{\boldsymbol{X}}_{k+1 / k+1} & =\hat{\boldsymbol{X}}_{k+1 / k}+\boldsymbol{K}_{k+1}\left(\boldsymbol{Z}_{k+1}-\boldsymbol{H}_{k+1 / k}\left(\hat{\boldsymbol{X}}_{k+1 / k}\right)\right),  \tag{8.27}\\
\boldsymbol{P}_{k+1} & =\left(\boldsymbol{I}-\boldsymbol{K}_{k+1} \boldsymbol{H}_{k+1 / k}\right) \boldsymbol{P}_{k+1 / k} \tag{8.28}
\end{align*}
$$

### 8.2.3 Mathematical Model Considering $J_{2}$ Perturbation of Earth Oblateness

When only a two-body problem is being considered, the difference between the result of orbit extrapolation and the actual orbit (which is influenced by various perturbative forces) increases rapidly with the increase in extrapolation time. Take the LEO (low earth orbit) satellite with an orbital altitude of 800 km as an example; an extrapolation error of around 1000 seconds can be up to an order of 20 km when the two-body model is directly adopted for orbit extrapolation. While the $J_{2}$ perturbation system model is adopted for extrapolation, such an error of around 1000 seconds is on an order of 100 m . Therefore, to analyze the passive tracking performance of satellite-to-satellite tracking seems too idealized. The orbit perturbation influence will be considered based on the two-body model in this section. For the purpose of analysis, here the influence from $J_{2}$ perturbation of the earth's oblateness, which has the largest influence, is preferably under consideration, and the analysis and solutions are given. Such approximation for satellite-to-satellite bearings-only passive tracking may be accurate enough in most applications. Based on such an analysis, add the partial derivative of the corresponding perturbative force on the position vector.
The position and velocity vectors of the target satellite in the J2000.0 coordinate system are taken as the state variable $\boldsymbol{X}$, as shown in expression (8.1). Using expression (8.1), the state differential equation for satellite motion considering $J_{2}$ perturbation can be expressed as

$$
\dot{\boldsymbol{X}}=\overline{\boldsymbol{F}}(\boldsymbol{X})=\left[\begin{array}{ll}
\dot{\boldsymbol{r}}^{\mathrm{T}} & \ddot{\boldsymbol{r}}^{\mathrm{T}} \tag{8.29}
\end{array}\right]^{\mathrm{T}} .
$$

To find the difference between results in the case of the two-body model, here $\overline{\boldsymbol{F}}(\cdot)$ is the nonlinear transformation against the state variable. From reference [4], the perturbed motion equation of the satellite in the J2000.0 coordinate system can be expressed as

$$
\left.\begin{array}{rl}
\ddot{x} & =-\frac{\mu}{r^{3}} x+\frac{\partial R}{\partial x} \\
\ddot{y} & =-\frac{\mu}{r^{3}} y+\frac{\partial R}{\partial y}  \tag{8.30}\\
\ddot{z} & =-\frac{\mu}{r^{3}} z+\frac{\partial R}{\partial z}
\end{array}\right\},
$$

where $R$ is the perturbation function, which mainly refers to the corresponding perturbation function of $J_{2}$ perturbation. If other perturbative forces are to be added, we just need to expand the perturbation function $R$. Obviously, when $R=0$, the above expression becomes the motion equation of the two-body model.
After derivation and simplification, the equation of satellite motion considering $J_{2}$ perturbation can be expressed as

$$
\begin{align*}
& \ddot{x}=-\frac{\mu}{r^{3}} x\left[1-J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(7.5 \frac{z^{2}}{r^{2}}-1.5\right)\right] \\
& \left.\ddot{y}=-\frac{\mu}{r^{3}} y\left[1-J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(7.5 \frac{z^{2}}{r^{2}}-1.5\right)\right]\right\} .  \tag{8.31}\\
& \ddot{z}=-\frac{\mu}{r^{3}} z\left[1-J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(7.5 \frac{z^{2}}{r^{2}}-4.5\right)\right]
\end{align*}
$$

The differential equation of the satellite motion state considering term $J_{2}$ can be obtained by substituting the above expression into expression (8.29).
As the orbit perturbation does not correlate with the observations, the measurement model considering $J_{2}$ perturbation of the earth oblateness is the same as that in the case of the two-body model.

### 8.2.4 Tracking Method Considering $\boldsymbol{J}_{2}$ Perturbation of Earth Oblateness

When considering $J_{2}$ perturbation of earth oblateness, only the differential equation of the satellite motion state is different from that in the case of the two-body model, so the corresponding state prediction equation and state transition matrix are also different from those of the two-body model. The following is the derivation of differences from the results in the case of the two-body model.
Similarly, after discretizing the state differential equation (Equation (8.29)) for the target satellite, it can be found that

$$
\begin{equation*}
\boldsymbol{X}(k+1)-\boldsymbol{X}(k)=\int_{t_{k}}^{t_{k+1}} \overline{\boldsymbol{F}}(\boldsymbol{X}(t)) \mathrm{d} t \tag{8.32}
\end{equation*}
$$

When the time interval $t_{k+1}-t_{k}=T$ is short enough, $\overline{\boldsymbol{F}}(\boldsymbol{X}(t))$ can be expanded as the Taylor series with time $t_{k}$ :

$$
\begin{equation*}
\overline{\boldsymbol{F}}(\boldsymbol{X}(t)) \approx \overline{\boldsymbol{F}}(\boldsymbol{X}(k))+\overline{\boldsymbol{A}}(\boldsymbol{X}(k)) \cdot \overline{\boldsymbol{F}}(\boldsymbol{X}(k))\left(t-t_{k}\right) \tag{8.33}
\end{equation*}
$$

where

$$
\overline{\boldsymbol{A}}(\boldsymbol{X}(k))=\left.\frac{\partial \overline{\boldsymbol{F}}(\boldsymbol{X}(t))}{\partial \boldsymbol{X}}\right|_{t=t_{k}}
$$

is a $6 \times 6$ matrix. Substitute expression (8.33) into expression (8.32) to obtain

$$
\begin{equation*}
\boldsymbol{X}(k+1)=\boldsymbol{X}(k)+\overline{\boldsymbol{F}}(\boldsymbol{X}(k)) T+\overline{\boldsymbol{A}}(\boldsymbol{X}(k)) \cdot \overline{\boldsymbol{F}}(\boldsymbol{X}(k)) \frac{T^{2}}{2}+\overline{\boldsymbol{W}}(k), \tag{8.34}
\end{equation*}
$$

where $\bar{W}(k)$ is the error vector introduced when the nonlinear state equation is linearized. This is in a different order from $\boldsymbol{W}(k)$, for $\overline{\boldsymbol{W}}(k)$ is in the same order as $|\boldsymbol{X}(k)-\widehat{\boldsymbol{X}}(k / k)|^{2}$. Suppose that $\mathrm{E}\left[\overline{\boldsymbol{W}}(k) \overline{\boldsymbol{W}}(k)^{\mathrm{T}}\right]=\overline{\boldsymbol{Q}}$ is the state noise covariance matrix.
From expression (8.29) and the vector differentiation law [5], $\overline{\boldsymbol{A}}(\boldsymbol{X}(k))$ can be expressed as follows:

$$
\overline{\boldsymbol{A}}(\boldsymbol{X}(k))=\left[\begin{array}{cc}
\frac{\partial \dot{\boldsymbol{r}}}{\partial \boldsymbol{r}_{3 \times 3}} & \frac{\partial \dot{\boldsymbol{r}}}{\partial \dot{\boldsymbol{r}}_{3 \times 3}}  \tag{8.35}\\
\frac{\partial \ddot{\boldsymbol{r}}}{\partial \boldsymbol{r}_{3 \times 3}} & \frac{\partial \ddot{\boldsymbol{r}}}{\partial \dot{\boldsymbol{r}}_{3 \times 3}}
\end{array}\right]=\left[\begin{array}{cc}
0_{3 \times 3} & \boldsymbol{I}_{3 \times 3} \\
\frac{\partial \dot{\boldsymbol{r}}}{\partial \boldsymbol{r}_{3 \times 3}} & 0_{3 \times 3}
\end{array}\right]_{t=t_{k}}
$$

where $\boldsymbol{r}, \dot{\boldsymbol{r}}$, and $\ddot{\boldsymbol{r}}$ are respectively the position, velocity, and acceleration vectors of the target satellite in the J2000.0 coordinate system and $\dot{\boldsymbol{j}} / \partial r$ can be solved by expressions (8.31) and (8). After simplification, it can be found that

$$
\frac{\partial \ddot{\boldsymbol{r}}}{\partial \boldsymbol{r}}=\frac{\mu}{r^{3}} \boldsymbol{B}=\frac{\mu}{r^{3}}\left[\begin{array}{lll}
B_{11} & B_{12} & B_{13}  \tag{8.36}\\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{array}\right],
$$

where the components in matrix $\boldsymbol{B}$ are expressed as follows:

$$
\begin{align*}
& B_{11}=\frac{3 x^{2}}{r^{2}}-1-\frac{3 x^{2}}{r^{2}} J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(17.5 \frac{z^{2}}{r^{2}}-2.5\right)+J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(7.5 \frac{z^{2}}{r^{2}}-1.5\right) \\
& B_{12}=\frac{3 x y}{r^{2}}-\frac{3 x y}{r^{2}} J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(17.5 \frac{z^{2}}{r^{2}}-2.5\right) \\
& B_{13}=\frac{3 x z}{r^{2}}-\frac{3 x z}{r^{2}} J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(17.5 \frac{z^{2}}{r^{2}}-7.5\right) \\
& B_{22}=\frac{3 y^{2}}{r^{2}}-1-\frac{3 y^{2}}{r^{2}} J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(17.5 \frac{z^{2}}{r^{2}}-2.5\right)+J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(7.5 \frac{z^{2}}{r^{2}}-1.5\right)  \tag{8.37}\\
& B_{23}=\frac{3 y z}{r^{2}}-\frac{3 y z}{r^{2}} J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(17.5 \frac{z^{2}}{r^{2}}-7.5\right) \\
& B_{33}=\frac{3 z^{2}}{r^{2}}-1-\frac{3 z^{2}}{r^{2}} J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(17.5 \frac{z^{2}}{r^{2}}-12.5\right)+J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(7.5 \frac{z^{2}}{r^{2}}-4.5\right) \\
& B_{21}=B_{12}, B_{31}=B_{13}, B_{32}=B_{23}
\end{align*}
$$

Obviously, matrix $\mathbf{B}$ considering term $J_{2}$ perturbation is much more complicated than that in the case of the two-body model, but now matrix $\mathbf{B}$ is still symmetric. Substituting expressions (8.36) and (8.37) into expression (8.35), $\overline{\mathbf{A}}(\mathbf{X}(k))$ can be obtained.

Substitute the filtered state $\widehat{\mathbf{X}}(k / k)$ into expressions (8.35) and (8.34), respectively, and the state prediction equation considering $J_{2}$ perturbation of earth oblateness can be expressed as

$$
\begin{equation*}
\widehat{\boldsymbol{X}}(k+1 / k)=\widehat{\boldsymbol{X}}(k / k)=\overline{\boldsymbol{F}}(\hat{\boldsymbol{X}}(k / k)) T+\overline{\boldsymbol{A}}(\hat{\boldsymbol{X}}(k / k)) \cdot \overline{\boldsymbol{F}}(\hat{\boldsymbol{X}}(k / k)) \frac{T^{2}}{2} \tag{8.38}
\end{equation*}
$$

The state transition matrix can be calculated as a method similar to that in the case of the two-body model. Firstly, the continuous state transition matrix is obtained. Then expand it as the Taylor series with time $t_{k}$. Finally, discretize it. The state transition matrix after discretization is noted to be $\overline{\boldsymbol{\Phi}}(k+1 / k)$ and its expression is as follows:

$$
\begin{equation*}
\overline{\boldsymbol{\Phi}}(k+1 / k)=\boldsymbol{I}+\overline{\boldsymbol{A}}(\hat{\boldsymbol{X}}(k / k)) T \tag{8.39}
\end{equation*}
$$

The state noise covariance matrix $\overline{\boldsymbol{Q}}$ considering $J_{2}$ perturbation can be selected as a small constant matrix by experience to reduce the computational load [2].

### 8.3 System Observability Analysis

### 8.3.1 Description Method for System Observability

Methods for measuring the system observability include the condition number of the observable matrix [6-8] and the method for adopting the eigenvalue and eigenvector [9]. Generally speaking, it is not easy to get an analytic solution of the eigenvalue and eigenvector, and the system observability obtained by the above measurement methods cannot show the influence of measurement noise. Therefore, it is necessary to find a kind of system observability description method that can better reflect the measured noise.
By the theorem of observability in Section 2.8, for a nonlinear equation set $\boldsymbol{Z}=\boldsymbol{h}\left(\boldsymbol{X}_{k}\right)$, the approximate first-order Newton iteration solution is

$$
\begin{equation*}
\boldsymbol{X}_{k}=\boldsymbol{X}_{k}^{*}+\boldsymbol{\Gamma}^{-1}(k-N+1, k)\left[\boldsymbol{Z}-\boldsymbol{h}\left(\boldsymbol{X}_{k}^{*}\right)\right], \tag{8.40}
\end{equation*}
$$

where $\boldsymbol{Z}=\left[\boldsymbol{Z}_{k-N+1}^{T}, \ldots, \boldsymbol{Z}_{k}^{T}\right]^{T}, \boldsymbol{h}\left(\boldsymbol{X}_{k}\right)=\left[\boldsymbol{h}_{k-N+1}^{T}\left(\boldsymbol{X}_{k-N+1}\right), \ldots, \boldsymbol{h}_{k}^{T}\left(\boldsymbol{X}_{k}\right)\right]^{T}$, and $\boldsymbol{X}_{k}^{*}$ is the initial state of $\boldsymbol{X}_{k}$, so the observability of the system can be reflected in whether $\boldsymbol{\Gamma}(k-N+1, k)$ is approaching singularity or pathosis. If $\boldsymbol{X}_{k}^{*}$ is the true value, we can tell from expression (8.40) that $\boldsymbol{\Gamma}^{-1}(k-N+1, k)$ reflects the sensitivity that error $\boldsymbol{X}_{k}-\boldsymbol{X}_{k}^{*}$ of the solution is influenced by measured noise $\boldsymbol{Z}-\boldsymbol{h}\left(\boldsymbol{X}_{k}^{*}\right)$. If the covariance matrix of measured noise is denoted as $\boldsymbol{R}$, the state covariance matrix can be expressed as

$$
\begin{equation*}
\operatorname{cov}\left(\boldsymbol{X}_{k}-\boldsymbol{X}_{k}^{*}\right)=\left(\boldsymbol{\Gamma}^{\boldsymbol{T}} \boldsymbol{R}^{-1} \boldsymbol{\Gamma}\right)^{-1} \tag{8.41}
\end{equation*}
$$

As $\operatorname{det}\left(\left(\boldsymbol{\Gamma}^{\boldsymbol{T}} \boldsymbol{R}^{-1} \boldsymbol{\Gamma}\right)^{-1}\right)$ is the measurement of uncertainty hyper-ellipsoid volume under a Gaussian condition, here it can be introduced as a measurement of observability of the system, that is, the observability of the system $\rho$ is [10]

$$
\begin{equation*}
\rho=\sqrt{\left|\operatorname{det}\left(\left(\boldsymbol{\Gamma}^{T} \boldsymbol{R}^{-1} \boldsymbol{\Gamma}\right)^{-1}\right)\right|} \tag{8.42}
\end{equation*}
$$

There are two advantages to adopting $\rho$ for describing system observability:

1. It can reflect the relationship between the system observability and the measured noise. As the covariance matrix $\boldsymbol{R}$ of measured noise is needed for calculating $\rho$, such a description method can reflect the relationship between the system observability and the measurement equation.
2. It is easy for calculation. System observability $\rho$ is a scalar, which is easy for calculation and comparison.

A set of tracking scenarios are taken as examples for simulations as follows. Analyze the correlated influencing factors between computer simulation results of observability of the satellite passive tracking system by bearings-only and state equation/measurement equations. Here the ephemeris of two satellites in the J2000.0 coordinate system are generated by simulations based on orbit elements set by $\mathrm{STK}^{\circledR} 6.0$ software, where the filtering period is $T$, the observation duration is 5000 seconds, and the angle measurement error $\sigma_{\beta}=\sigma_{\varepsilon}$. See Table 8.1 for orbit elements of two satellites for analysis on system observability.

### 8.3.2 Influence of Factors on the State Equation

See expression (8.2) for the state equation of the satellite-to-satellite passive tracking system with bearings only. In expression (8.2), the state transition matrix after discretization is expressed as $\boldsymbol{\Phi}_{k+1, k}=\boldsymbol{I}+\boldsymbol{A}(\hat{\boldsymbol{X}}(k / k)) T$. As $\boldsymbol{\Phi}_{k+1, k}$ is basically determined by filtering period $T$ and changes slightly with variation of satellite position parameter, here the analysis on the relationship between the state equation and system observability focuses on the influence of the filtering period $T$ on system observability.
Assuming that the angle measurement error $\sigma_{\beta}=20^{\prime \prime}$, the filtering periods $T=5$ seconds, $T=20$ seconds, and $T=50$ seconds are selected to test the system observability variation curves obtained according to position and velocity estimation errors and expression (8.42). In addition, a method [8] adopting the condition number for indicating system observability is simulated. See Figure 8.2 for the observability curve, position, and velocity estimation error curves (if not specially noted, the position and velocity estimation errors in a system observability analysis all refer to the RMS (root mean square) error of the Monte Carlo simulation repeated 50 times).

Table 8.1 Orbit elements of two satellites in localization observability analysis

| Description of orbit elements | Orbit elements of <br> observing satellite | Orbit elements of <br> target satellite |
| :--- | :---: | :---: |
| Semi-major axis $a(\mathrm{~km})$ | 42000 | 7171 |
| Eccentricity $e$ | 0.1 | 0 |
| Inclination $i\left({ }^{\circ}\right)$ | 120 | 30 |
| RAAN $\Omega\left({ }^{\circ}\right)$ | 30 | 75 |
| Argument of perigee $\omega\left({ }^{\circ}\right)$ | 45 | 60 |
| Epoch mean argument of perigee $M_{0}\left({ }^{\circ}\right)$ | 0 | 30 |



Figure 8.2 Relationship between observability, position, and velocity estimation error curves and filtering periods. (a) Observability curve indicated by condition number, (b) observability curve obtained according to expression (8.42), (c) position estimation error curve, and (d) velocity estimation error curve

From Figure 8.2 it can be seen that the shorter the filtering period $T$, the smaller is the system observability value and the better the system observability, the higher is the corresponding position and velocity estimation precision; conversely, the longer the filtering period $T$, the worse is the system observability and the lower is the corresponding position and velocity estimation precision. In other words, filtering period $T$ is an important parameter, which influences the passive tracking system by bearings only and directly determines the state transition matrix $\boldsymbol{\Phi}_{k+1, k}$. As a result, it seriously influences the system observability.

### 8.3.3 Influence of Factors on the Measurement Equation

See expression (8.6) for the measurement equation of the satellite-to-satellite passive tracking system by bearings only. The corresponding Jacobian matrix is expressed as expression (8.21). Here analysis on the relationship between the measurement equation and system observability


Figure 8.3 Relationship between observability, position, and velocity estimation error curves and angle measurement errors. (a) Observability curve indicated with condition number, (b) observability curve obtained according to expression (8.42), (c) position estimation error curve, and (d) velocity estimation error curve
focuses on the influence of the angle measurement error $\sigma_{\beta}$ on system observability. Now, assuming that the filtering periods are the same, $T=5$ seconds, test the system observation variation curves of the position and velocity estimation error under different angle measurement errors, $\sigma_{\beta}=10^{\prime \prime}, \sigma_{\beta}=50^{\prime \prime}, \sigma_{\beta}=150^{\prime \prime}$, and expression (8.42). Likewise, here a method adopting the condition number for indicating system observations [8] is introduced as a comparison. See Figure 8.3 for the observability curve and position and velocity error curves.
According to Figure 8.3, the smaller the angle measurement error, the smaller is the system observability value and the better the system observability, the higher is the corresponding position and velocity estimation precision; conversely, the better the angle measurement error, the worse is the system observability and the lower the corresponding position and velocity estimation precision. In other words, for the satellite passive tracking system with bearings only, the angle measurement error is also an important parameter, which influences the system observability.

From Figure 8.3a and b , the method adopting a condition number for indicating system observability cannot reflect the influence of the angle measurement error on system observability. Therefore, with different measurement errors, the observability curves indicated with a condition number are the same, while system observability calculated according to expression (8.42) can clearly reflect the relationship between it and the angle measurement error. As the angle measurement error increases, the system observability value increases in an obvious manner, and the system observability becomes worse.

### 8.4 Tracking Simulation and Analysis

In this section, satellite-to-satellite passive tracking with bearings only, mentioned in Section 8.2 under different simulation conditions, is simulated and analyzed. As in the classification described in Section 8.2, the system model will be considered in the case of the two-body model and $J_{2}$ perturbation of the earth oblateness. Here, the CRLB (Cramér-Rao lower bound) under corresponding conditions is simulated for assessing the performance of the passive tracking system, verifying whether the algorithm is optimal and how much room is left for improvement.
Factors that influence the performance of the nonlinear filter usually include the measurement error, initial state error, target model error, and so on. In this section, passive tracking with bearings only with regards to the initial state error, angle measurement error, filtering period, and ephemeris error of the observer itself is simulated. The influence of various factors on tracking precision is analyzed, and further corresponding conclusions are given.
For passive tracking of theLEO satellite, in order to get observation arcs long enough, it is acceptable to locate the observing satellite on the geosynchronous orbit and use the high earth orbit (HEO) satellite to observe the LEO satellite. This will be the typical geometric scenario setting for simulations in this section. The observability is good when the orbital plane of the observing satellite is nearly perpendicular to that of the target satellite (please refer to the analysis in Section 8.3), so here the example is still the simulation scenario shown in Table 8.1. The ephemeris of two satellites in the J2000.0 coordinate system are generated through simulations based on orbit elements set by $\mathrm{STK}^{\circledR} 6.0$ software under the two-body model and considering the $J_{2}$ perturbation model of the earth, respectively. They can be processed with the EKF under corresponding situations where the filtering period is $T$ and the observation duration is 3000 seconds, assuming the angle measurement error $\sigma_{\beta}=\sigma_{\varepsilon}$.
Using the 'Access' function of satellite-to-satellite tracking in STK ${ }^{\circledR} 6.0$ software, a table of observable periods between two satellites within one day in the simulation can be generated, as shown in Table 8.2. The conclusion that the minimum period (i.e., not shielded by the earth) that the observing satellite is visible to the target satellite is longer than 3600 seconds and the maximum period is longer than 14000 seconds, along with an average observable duration of about 5000 seconds, can be drawn from Table 8.2. Therefore, if a bearings-only tracking algorithm is possible to converge within the minimum observable period, there is no need to take the shielding effect of the earth into consideration in the case of observing the satellite-to-target satellite passive orbit determination tracking.
In STK ${ }^{\circledR} 6.0$ software, it is possible to output a 2 D image (i.e., subsatellite track) and a 3D image of relative geometric relations between the satellites in such a simulation scenario, as shown in Figures 8.4 and 8.5, respectively.

Table 8.2 Observable period of the observing satellite against the target satellite

| Access | Start time | Stop time | Duration (s) |
| :--- | :--- | :--- | ---: |
| 1 | 1 January 2001 00:16:52.25 | 1 January 2001 02:37:57.45 | 8465.202 |
| 2 | 1 January 2001 03:04:05.53 | 1 January 2001 04:09:34.55 | 3929.014 |
| 3 | 1 January 2001 04:46:41.38 | 1 January 2001 05:47:46.84 | 3665.456 |
| 4 | 1 January 2001 06:27:08.75 | 1 January 2001 07:27:37.79 | 3629.042 |
| 5 | 1 January 2001 08:06:43.19 | 1 January 2001 09:08:23.74 | 3700.553 |
| 6 | 1 January 2001 09:45:04.89 | 1 January 2001 10:50:34.59 | 3929.702 |
| 7 | 1 January 2001 11:20:13.99 | 1 January 2001 15:20:00.75 | 14386.767 |
| 8 | 1 January 2001 15:35:56.83 | 1 January 2001 16:47:27.28 | 4290.443 |
| 9 | 1 January 2001 17:21:33.71 | 1 January 2001 18:24:08.49 | 3754.780 |
| 10 | 1 January 2001 19:02:53.09 | 1 January 2001 20:03:05.99 | 3612.895 |
| 11 | 1 January 2001 20:42:54.44 | 1 January 2001 21:43:18.12 | 3623.678 |
| 12 | 1 January 2001 22:21:03.38 | 1 January 2001 23:26:02.37 | 3898.989 |



Figure 8.4 2D subsatellite track of two satellites in the simulation scenario


Figure 8.5 3D relative geometric relation between two satellites in the simulation scenario

### 8.4.1 Simulation in the Case of the Two-Body Model

### 8.4.1.1 Simulation 1: Influences of State Initialization on Passive Tracking Performance

Assuming the angle measurement $\sigma_{\beta}=20^{\prime \prime}$ and the filtering period $T=2$ seconds, select three groups of different initial state errors of $\left[\begin{array}{llllll}100 & 100 & 100 & 0.1 & 0.1 & 0.1\end{array}\right]^{\mathrm{T}}$ (Case 1), $\left[\begin{array}{lllllll}300 & 300 & 300 & 0.5 & 0.5 & 0.5\end{array}\right]^{\mathrm{T}}$ (Case 2), and $\left[\begin{array}{lllll}500 & 500 & 500 & 1 & 1\end{array} 1\right]^{\mathrm{T}}$ (Case 3), where the units of initial state errors are kilometer for position vector $\boldsymbol{r}$ and kilometers per second for velocity vector $\dot{\boldsymbol{r}}$. The method in reference [1] is also simulated for comparison, of which the initial state error is $\left[\begin{array}{lllll}200 & 0.0005 & 3 & 3 & 3\end{array} 0\right.$ 畐 (Case 4) (see Table 8.1 for its units). The CRLBs of the same angle measurement accuracy are the same. See Figure 8.6 for satellite tracking and the azimuth angle, elevation angle, and relative distance variation curves in Case 2 (in the figure, the HEO satellite is the observer and the LEO satellite is the target).
According to Figure 8.6a, the initialization error of the target satellite position is large at the starting stage of bearings-only tracking. After a short time of state update, however, such a method can achieve good performance for tracking the target satellite. The estimated track obtained in this way can basically match that of the actual track of the target satellite.


Figure 8.6 Satellite track and azimuth angle, elevation angle, and relative distance curves. (a) Passive satellite tracking geometry, (b) azimuth angle curve, (c) elevation angle curve, and (d) relative distance curve


Figure 8.7 Position and velocity error curves of different initial state errors in the case of the two-body model. (a) Estimated position error curve and (b) estimated velocity error curve

Based on the estimated position and velocity vector of each point, it is possible to calculate the position and velocity errors of the target satellite, as shown in Figure 8.7 (if not specially noted again in this book, for passive tracking performance simulation, the position and velocity estimation errors both refer to RMS errors of Monte Carlo simulations repeated 100 times).
The converged position and velocity estimation errors and corresponding CRLB under the above conditions are shown in Table 8.3. According to Figure 8.7 and Table 8.3, the bearings-only tracking method proposed in this chapter can adapt to various initial state errors and, finally, converges stably in the case of the two-body model. Although when the initial state error increases, the time required by filter convergence and the final estimation error both increase in a certain way, the performance of the algorithm can still approach that of the CRLBs. While using the method described in references [1] and [11], the target satellite will be confronted with a singularity issue when the typical orbit elements are used as the state

Table 8.3 Position and velocity errors of different initial state errors in the case of the two-body model

| Simulation scenario/parameter | Case 1 | Case 2 | Case 3 | Case 4 | CRLB |
| :--- | ---: | :--- | :--- | ---: | :---: |
| Position estimation error $(\mathrm{km})$ | 1.0566 | 1.1298 | 1.4275 | 31.7958 | 0.6489 |
| Velocity estimation error $(\mathrm{km} / \mathrm{s})$ | 0.0012 | 0.0012 | 0.0013 | 0.0155 | $7.1867 \times 10^{-4}$ |



Figure 8.8 Position and velocity error curves in the case of the two-body model with an angle measurement error $\sigma_{\beta}=10^{\prime \prime}$. (a) Estimated position error curve and (b) estimated velocity error curve
variable. This will have a certain influence on its filter performance and, as a result, the time required by filter convergence and estimation error will increase dramatically compared with the bearings-only tracking method shown in this chapter.

### 8.4.1.2 Simulation 2: Influences of Different Angle Measurement Errors on Passive Tracking Performance

Assuming the filtering period $T=2$ seconds and the initial state error is $\left[\begin{array}{lll}300 & 300 & 300 \\ 0.5 & 0.5\end{array}\right.$ $0.5]^{\mathrm{T}}$, calculate the estimation error of passive tracking when angle measurement errors are $10^{\prime \prime}, 50^{\prime \prime}, 150^{\prime \prime}$, and $200^{\prime \prime}$, respectively. As CRLBs are different with different angle measurement errors, the position and velocity estimation error curves of the target satellite with different angle measurement errors are now given:


Figure 8.9 Position and velocity error curves in the case of the two-body model with an angle measurement error $\sigma_{\beta}=50^{\prime \prime}$. (a) Estimated position error curve and (b) estimated velocity error curve.


Figure 8.10 Position and velocity error curves in the case of the two-body model with an angle measurement error $\sigma_{\beta}=150^{\prime \prime}$. (a) Estimated position error curve and (b) estimated velocity error curve

1. When the angle measurement error $\sigma_{\beta}=10^{\prime \prime}$, the simulation results are as shown in Figure 8.8.
2. When the angle measurement error $\sigma_{\beta}=50^{\prime \prime}$, the simulation results are as shown in Figure 8.9.
3. When the angle measurement error $\sigma_{\beta}=150^{\prime \prime}$, the simulation results are as shown in Figure 8.10.
4. When the angle measurement error $\sigma_{\beta}=200^{\prime \prime}$, the simulation results are as shown in Figure 8.11.

The converged position and velocity estimation errors and corresponding CRLBs under the prescribed four groups of simulation conditions are shown in Table 8.4.


Figure 8.11 Position and velocity error curves in the case of the two-body model with an angle measurement error $\sigma_{\beta}=200^{\prime \prime}$. (a) Estimated position error curve and (b) estimated velocity error curve

According to Figures 8.8 to 8.11 and Table 8.4, in the case of the two-body model, with angle measurement error increases, the filter convergence time, position, and velocity estimation errors of the proposed bearings-only tracking method all increase dramatically. When the angle measurement error $\sigma_{\beta}=200^{\prime \prime}$, the position estimation error is nearly up to 10 km , which indicates that the bearings-only tracking method is very sensitive to the angle measurement error. Additionally, when the angle measurement error is less than $200^{\prime \prime}$, the position and velocity estimation error curves are both close to the corresponding CRLB curve, which indicates that when the angle measurement error is small in the case of the two-body model, the tracking algorithm is close to optimal.

Table 8.4 Position and velocity estimation errors of different angle measurement errors in the case of the two-body model

| Simulation scenario/parameter | $\sigma_{\beta}=10^{\prime \prime}$ |  | $\sigma_{\beta}=50^{\prime \prime}$ |  |
| :---: | :---: | :---: | :---: | :---: |
|  | Estimation error | CRLB | Estimation error | CRLB |
| Position estimation error (km) | 0.9114 | 0.3245 | 1.8138 | 1.6223 |
| Velocity estimation error (km/s) | $9.5498 \times 10^{-4}$ | $3.5934 \times 10^{-4}$ | 0.0020 | 0.0018 |
| Simulation scenario/parameter | $\sigma_{\beta}=150^{\prime \prime}$ |  | $\sigma_{\beta}=200^{\prime \prime}$ |  |
|  | Estimation error | CRLB | Estimation error | CRLB |
| Position estimation error (km) | 4.8663 | 4.2059 | 9.5796 | 6.4889 |
| Velocity estimation error (km/s) | 0.0054 | 0.0050 | 0.0105 | 0.0072 |

### 8.4.1.3 Simulation 3: Influences of Different Filtering Periods on Passive Tracking Performances

Assuming the initial state error is still $\left[\begin{array}{lllll}300 & 300 & 300 & 0.5 & 0.5\end{array} 0.5\right]^{\mathrm{T}}$ and the angle measurement error $\sigma_{\beta}=20^{\prime \prime}$, calculate the passive location tracking estimation error in the case of the filtering periods $T=5$ seconds and $T=50$ seconds. As CRLBs are different in different filtering periods, here the estimation error curves under different filtering periods are given:

1. When the filtering period $T=5$ seconds, the simulation results are as shown in Figure 8.12.
2. When the filtering period $T=50$ seconds, the simulation results are as shown in Figure 8.13.

The converged position and velocity estimation errors and corresponding CRLBs under the two groups of simulation conditions are shown in Table 8.5.
According to Figures 8.4 to 8.13 and Table 8.5, in the case of the two-body problem, with filtering period increases, the filter convergence time, position, and velocity estimation errors of the proposed bearings-only tracking method discussed in this chapter all increase dramatically. When the filtering period is 50 seconds, the position estimation error is up to 16 km and its curve is obviously different from its corresponding CRLB curve, which indicates that the bearings-only tracking method is also very sensitive to the filtering period. In the case of the two-body problem, as there is a linearization error when using the EKF, with filtering period increases the linearization error increases, leading to worse tracking filter precision.

Table 8.5 Position and velocity estimation errors of different filtering periods in the case of the two-body problem

| Simulation scenario/parameter | $T=5 \mathrm{~s}$ |  | $T=50 \mathrm{~s}$ |  |
| :---: | :---: | :---: | :---: | :---: |
|  | Estimation error | CRLB | Estimation error | CRLB |
| Position estimation error (km) | 1.1023 | 0.7084 | 16.3420 | 3.4958 |
| Velocity estimation error (km/s) | 0.0013 | $5.8446 \times 10^{-4}$ | 0.0133 | 0.0036 |



Figure 8.12 Position and velocity error curves in the case of the two-body model with a filtering period $T=5$ seconds. (a) Estimated position error curve and (b) estimated velocity error curve

### 8.4.1.4 Simulation 4: Influences of the Ephemeris Error of Observing Satellite on Passive Tracking Performance

In this chapter, for simulations 1 to 3 , it is assumed that there is no error in the ephemeris of the observing satellite. In fact, it is not possible for the ephemeris of the observer itself to be precisely accurate. This simulation analysis will be done in simulation 4. Assuming that the initial state error is still $\left[\begin{array}{lllll}300 & 300 & 300 & 0.5 & 0.5 \\ 0.5\end{array}\right]^{\mathrm{T}}$, the angle measurement error $\sigma_{\beta}=20^{\prime \prime}$, and the filtering period $T=10$ seconds, calculate the passive tracking estimation errors for the three cases: Case 1, where there is no error in the ephemeris of the observer, Case 2 with position error of 200 m and velocity error of $10 \mathrm{~m} / \mathrm{s}$, and Case 3 with position error of 500 m


Figure 8.13 Position and velocity error curves in the case of the two-body model with a filtering period $T=50$ seconds. (a) Estimated position error curve and (b) estimated velocity error curve
and velocity error of $50 \mathrm{~m} / \mathrm{s}$. The position and velocity estimation error curves based on the above cases are shown in Figure 8.14. The converged position and velocity estimation errors with the above conditions are shown in Table 8.6.
From Figure 8.14 and Table 8.6, for the case of the two-body model with the ephemeris error of the observer itself under consideration, there is no obvious difference from the position and velocity estimation errors by the bearings-only tracking method in this chapter, in which the position error varies around 100 m and the variation of velocity error is less than $1 \mathrm{~m} / \mathrm{s}$. This indicates that the ephemeris error of the observer itself is an influencing factor that can be ignored in satellite passive orbit determination tracking.


Figure 8.14 Position and velocity estimation error curves of the observer with different ephemeris errors in the case of the two-body model. (a) Estimated position error curve and (b) estimated velocity error curve

Table 8.6 Position and velocity estimation errors of the observer with different ephemeris errors in the case of the two-body model

| Simulation scenario/parameter | Case 1 | Case 2 | Case 3 |
| :--- | :--- | :--- | :--- |
| Position estimation error $(\mathrm{km})$ | 1.3716 | 1.2439 | 1.4917 |
| Velocity estimation error $(\mathrm{km} / \mathrm{s})$ | 0.0015 | 0.0016 | 0.0017 |

### 8.4.2 Simulation Considering $J_{2}$ Perturbation of Earth Oblateness

As described above, passive tracking performance with bearings only is analyzed through simulation in the case of the two-body model. The following simulations are about passive tracking performance with bearings only considering $J_{2}$ perturbation of the earth oblateness.

### 8.4.2 Simulation 5: Influences of Different Initial State Errors on Passive Tracking Performance

Assume that the angle measurement error $\sigma_{\beta}=20^{\prime \prime}$, filtering period $T=2$ seconds, two groups of initial state error $\boldsymbol{X}_{e}=\left[\begin{array}{lllll}300 & 300 & 300 & 0.5 & 0.5 \\ 0.5\end{array}\right]^{\mathrm{T}}$ (Case 2), and $\boldsymbol{X}_{e}=\left[\begin{array}{lll}500 & 500\end{array}\right.$ $5001111]^{\mathrm{T}}$ (Case 3) are adopted considering $J_{2}$ perturbation, where the units of the initial state error are kilometer for position vector $\boldsymbol{r}$ and kilometers per second for velocity vector $\dot{\boldsymbol{r}}$. Additionally, the estimated result when the initial state error in the case of the two-body model is $\boldsymbol{X}_{e}=\left[\begin{array}{lllll}300 & 300 & 300 & 0.5 & 0.5 \\ 0.5\end{array}\right]^{\mathrm{T}}$ (Case 1) is compared with the CRLB (whether orbit perturbation is considered does not have much influence on CRLB, if not specially noted in this chapter, for the lower bound of estimation errors under the perturbation case all use CRLB in the case of the two-body model for the research). Based on the estimated position and velocity vectors of each point, it is possible to calculate the position and velocity estimation errors of the target satellite, as shown in Figure 8.15. The converged position and velocity estimation errors and corresponding CRLBs under the above conditions are shown in Table 8.7.
From Figure 8.15 and Table 8.7, the estimation error with perturbation considered is obviously larger than that in the case of the two-body model and there are differences in the CRLB. Furthermore, the filter is less stable than that in the case of the two-body model and the estimation error will oscillate within a certain scope, which indicates that the linearization error generated by the EKF with perturbation considered is more complicated than that in the case of the two-body model and that rectification with the noise covariance matrix in a constant state is not as effective. Additionally, with increases in the initial state error, the error of position and velocity estimation with the bearings-only tracking method discussed in this chapter increases in an apparent way.

### 8.4.2.2 Simulation 6: Influences of Different Angle Measurement Errors on Passive Tracking Performance

Assuming the filtering period $T=2$ seconds and the initial state error is $\boldsymbol{X}_{e}\left[\begin{array}{llll}300 & 300 & 300 & 0.5\end{array}\right]$ $0.50 .5]^{\mathrm{T}}$, calculate the passive tracking estimation error when angle measurement errors are $50^{\prime \prime}$ and $150^{\prime \prime}$, respectively, with perturbation considered, and where the CRLB is still introduced as the reference to the lower bound of the parameter estimation error:

1. When the angle measurement error $\sigma_{\beta}=50^{\prime \prime}$, the simulation results are as shown in Figure 8.16.
2. When the angle measurement error $\sigma_{\beta}=150^{\prime \prime}$, the simulation results are as shown in Figure 8.17.


Figure 8.15 Position and velocity estimation error curves of different initial state errors in the case of perturbation. (a) Estimated position error curve and (b) estimated velocity error curve

Table 8.7 Position and velocity estimation errors of different initial state errors in the case of perturbation

| Simulation scenario/parameter | Case 1 | Case 2 | Case 3 | CRLB |
| :--- | :--- | :--- | :--- | :--- |
| Position estimation error $(\mathrm{km})$ | 1.1298 | 2.6523 | 4.0558 | 0.6489 |
| Velocity estimation error $(\mathrm{km} / \mathrm{s})$ | 0.0012 | 0.0177 | 0.0179 | $7.1867 \times 10^{-4}$ |



Figure 8.16 Position and velocity estimation error curves in the case of perturbation with an angle measurement error $\sigma_{\beta}=50^{\prime \prime}$. (a) Estimated position error curve and (b) estimated velocity error curve

The converged position and velocity estimation errors and corresponding CRLB under the above two simulation conditions are shown in Table 8.8.
According to Figure 8.16, Figure 8.17, and Table 8.8, when perturbation is considered, with angle measurement error increases, the position and velocity estimation errors with the bearings-only tracking method increase in an apparent way. Compared with the corresponding results in the case of the two-body model in Simulation 2, either the position and velocity estimation precision or filter stability with perturbation considered is as good as those in the case of the two-body model.


Figure 8.17 Position and velocity estimation error curves in the case of perturbation with an angle measurement error $\sigma_{\beta}=150^{\prime \prime}$. (a) Estimated position error curve and (b) estimated velocity error curve

Table 8.8 Position and velocity estimation errors of different angle measurement errors in the case of perturbation

| Simulation scenario/parameter | $\sigma_{\beta}=50^{\prime \prime}$ |  |  | $\sigma_{\beta}=150^{\prime \prime}$ |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Estimation error | CRLB |  | Estimation error | CRLB |
| Position estimation error $(\mathrm{km})$ | 5.0624 | 1.6223 |  | 6.8377 | 4.2059 |
| Velocity estimation error $(\mathrm{km} / \mathrm{s})$ | 0.0185 | 0.0018 |  | 0.0209 | 0.0050 |



Figure 8.18 Position and velocity error curves in the case of perturbation when the filtering period $T=5$ seconds. (a) Estimated position error curve and (b) estimated velocity error curve

### 8.4.2 3 Simulation 7: Influences of Different Filtering Periods on Passive Tracking Performances

Assuming the initial state error is still $\left[\begin{array}{lllll}300 & 300 & 300 & 0.5 & 0.5\end{array} 0.5\right]^{\mathrm{T}}$ and the angle measurement error $\sigma_{\beta}=20^{\prime \prime}$, calculate the passive tracking estimation error when the filtering period $T=5$ seconds and $T=50$ seconds with perturbation considered. As the CRLBs are different in different filtering periods, here the estimation error curves in different filtering periods are given:

1. When the filtering period $T=5$ seconds, the simulation results are as shown in Figure 8.18.
2. When the filtering period $T=50$ seconds, the simulation results are as shown in Figure 8.19.


Figure 8.19 Position and velocity error curves in the case of perturbation when the filtering period $T=50$ seconds. (a) Estimated position error curve and (b) estimated velocity error curve

The converged position and velocity estimation errors and corresponding CRLB under the above two simulation conditions are shown in Table 8.9.
From Figure 8.18, Figure 8.19 and Table 8.9, with filtering period increases, the filter convergence time, position, and velocity estimation errors all increase in an obvious manner, which indicates that with perturbation considered, the bearings-only tracking method is still very sensitive to the filtering period. When the filtering period is 5 seconds, the estimated precision under perturbation is still not as good as that in the case of the two-body model, but when the filtering period increases to 50 seconds, the position estimation error under perturbation is 11.9 km , which is better than that in the case of the two-body model (at such a time, the position estimation error in the case of the two-body model is 16.3 km , as shown in Table 8.9). This indicates that, as with filtering period increases, the linearization error generated by the EKF algorithm with perturbation considered is more complicated than that in the case of the

Table 8.9 Position and velocity estimation errors of different filtering periods in the case of perturbation

| Simulation scenario/parameter | $T=5 \mathrm{~s}$ |  | $T=50 \mathrm{~s}$ |  |
| :--- | :---: | :--- | ---: | ---: |
|  | Estimation error | CRLB | Estimation errorCRLB |  |
| Position estimation error $(\mathrm{km})$ | 6.0135 | 0.7084 | 11.9188 | 3.4958 |
| Velocity estimation error $(\mathrm{km} / \mathrm{s})$ | 0.0185 | $5.8446 \times 10^{-4}$ | 0.0132 | 0.0036 |



Figure 8.20 Position and velocity estimation error curves of the observer with different ephemeris errors under perturbation. (a) Estimated position error curve and (b) estimated velocity error curve

Table 8.10 Position and velocity estimation error curves of the observer with different ephemeris errors under perturbation

| Simulation scenario/parameter | Case 1 | Case 2 | Case 3 |
| :--- | :--- | :--- | :--- |
| Position estimation error $(\mathrm{km})$ | 1.4917 | 6.0600 | 6.0866 |
| Velocity estimation error $(\mathrm{km} / \mathrm{s})$ | 0.0017 | 0.0193 | 0.0193 |

two-body model and that rectification with the noise covariance matrix in a constant state hardly satisfies the requirement of a high precision state estimation.

### 8.4.2.4 Simulation 8: Influences of the Ephemeris of Observing Satellite on Passive Tracking Performances

In this chapter, for simulations 5 to 7 , it is assumed that there is no error in the ephemeris of the observing satellite, for which simulation analysis will be done in Simulation 8. Assuming that the initial state error is still $\left[\begin{array}{lllll}300 & 300 & 300 & 0.5 & 0.5 \\ 0.5\end{array}\right]^{\mathrm{T}}$, the angle measurement error $\sigma_{\beta}=20^{\prime \prime}$, and the filtering period $T=10$ seconds, calculate the passive tracking estimation error under the three cases: position error of 500 m and velocity error of $50 \mathrm{~m} / \mathrm{s}$ (Case 1), for the ephemeris of the observing satellite in the case of the two-body model, no error in the ephemeris of the observing satellite (Case 2), position error of 500 m and velocity error of $50 \mathrm{~m} / \mathrm{s}$ (Case 3). See Figure 8.20 for the position and velocity estimation error curves based on the above cases. The converged position and velocity estimation errors under the above conditions are shown in Table 8.10.
According to Figure 8.20 and Table 8.10, when the ephemeris error of the observer itself is being considered, the position and velocity estimation errors under perturbation do not increase dramatically, but obviously are larger than those in the case of the two-body model, which indicates that the ephemeris error of the observer itself under perturbation is still an influencing factor, which can be ignored for satellite passive orbit determination tracking.

### 8.5 Summary

Through theoretical analysis and simulation results, the following conclusions can be achieved:

1. Under two-dimensional (two satellites on the same orbital plane) and three-dimensional conditions, the satellite-to-satellite passive tracking system with bearings only is totally observable. In other words, it is possible to achieve satellite-to-satellite passive tracking through several times of bearings observation.
2. Under typical simulation scenarios, what influences the estimation performance of passive tracking with bearings only most is the filtering period, the second is the angle measurement and initial state error, and the least is the ephemeris error of the observer itself. The estimation error of the algorithm increases with an increase in various influencing factors.
3. In the case of the two-body model only, the estimation performance of the bearings-only tracking method is very close to the lower bound of parameter estimation (CRLB), which
indicates that in the case of the two-body model, the EKF with bearings only mentioned in this chapter can achieve satellite-to-satellite passive tracking in an effective way.
4. In the case of perturbation, the estimation performance of the bearings-only tracking method is obviously worse than that in the case of the two-body model, along with differences in the CRLB, which indicates that under a complicated system model like perturbation, the linearization error generated by the EKF is much more complicated than that in the case of the two-body model, and that it s not enough to rectify the linearization error of the nonlinear state equation by a noise covariance matrix in the constant state only. Further research on the filter improvement algorithm under a complicated system model is needed for a state estimation result of high precision.

## References

1. Guo, F. and Fan, Y. (2005) A tracking method for satellite-to-satellite passive localization in space information confrontation. Journal of Astronautics, 26(2): P196-P200 (in Chinese).
2. Bar-Shalom, Y., Li, R.X. and Kjrubarajan, T. (2001) Estimation with Applications to Tracking and Navigation. Hoboken, NJ: John Wiley \& Sons, Inc.
3. Kay, S. (1998) Fundamentals of Statistical Signal Processing, Volume I: Estimation Theory/Volume II: Detection Theory, Prentice Hall PTR.
4. Xi, X., Wang, W., and Gao, Y. (2003) Fundamentals of Near-Earth Spacecraft Orbit. Changsha: NUDT Publish House (in Chinese).
5. Zhang, X. (2004) Matrix Analysis and Applications. Beijing: Tshinghua University Press, p. 9.
6. Ning, X. and Fang, J. (2005) A new autonomous celestial navigation method for deep space probe and its observability analysis. Chinese Journal of Space Science, 25(4): P286-P292 (in Chinese).
7. Ning, X. and Fang, J. (2005) Analysis of observability and the degree of observability of autonomous celestial navigation in spacecraft. Journal of Beijing University of Aeronautics and Astronautics, 31(6): P673-P677 (in Chinese).
8. Liu, Z. and Chen, Z. (2004) Application of condition number in observability analysis of system. Journal of System Simulation, 16(7): P1552-P1555 (in Chinese).
9. Ham, F.M. and Brown, R.G. (1983) Observability, eigenvalues, and Kalman filtering. IEEE Transactions on Aerospace and Electronic Systems, 19(2), 269-273.
10. Sun, Z., Zhou, Y., and He, L. (1996) Active and Passive Localization Technology with Single or Multiple Base. Beijing: National Defence Industry Press (in Chinese).
11. Guo, F. and Li, Q. (2005) Reconnaissance location technology of LEO satellites by MEO and HEO satellites. Proceedings of the 14th Annual Academic Conference on ECM of CIE, August, 2005, pp. P1531-P1535 (in Chinese).

## 9

## Satellite-to-Satellite Passive Tracking Based on Angle and Frequency Information

### 9.1 Introduction of Passive Tracking

Chan and Rudnicki [1] and Becker [2-4] have respectively pointed out that, when an angle is measured, the introduction of frequency measurement information can increase the observability of the passive localization system to reduce the filtering convergence time and improve localization accuracy. Since both measurement information of the angle and frequency are adopted, it is also called the combined method (CM) [2-4]. The question is whether the passive tracking system of the satellite-to-satellite target can introduce frequency measurement information.
According to Kepler's Third Law, the square of the running cycle of the satellite orbiting the earth is in proportion to the cube of the semi-major axis of its orbit [5, 6]. In other words, for two satellites running in different orbits, a relative motion must exist between the satellites. If the observing satellite can also measure frequency information of the arrived signal at the same time as finding measuring angle information, there may be some Doppler shift in the frequency of the received signal due to relative motion between the satellites, which is helpful when improving observability of the passive tracking system to reduce the filtering convergence time and to improve tracking accuracy.
Therefore, based on the study in Chapter 8, frequency measurement information is further introduced to study the satellite-to-satellite passive tracking method on the basis of angle and frequency information, establish such two passive tracking models as the two-body model and that considering the term $J_{2}$ perturbation of earth oblateness, and derive the corresponding extended Kalman filter (EKF) method when the angle is measured.
Moreover, this chapter will further study the observability of the satellite-to-satellite passive tracking system based on angle and frequency information and analyze relations between the system observability and the influencing factors correlated with the state equation and the measurement equation, which provide the basis for performance evaluation of the algorithm.

[^3]
### 9.2 Tracking Model and Method

Similar to research ideas in Chapter 8, we will start from the two-body model to establish the satellite-to-satellite passive tracking model based on angle and frequency information and then extend it to the condition considering $J_{2}$ perturbation of the earth oblateness and using the EKF method under corresponding conditions.

### 9.2.1 Mathematic Model in the Case of the Two-Body Model

### 9.2.1.1 State Model

Take the position and velocity vector and unknown signal carrier frequency $f_{0}$ of the target satellite in the J2000.0 coordinate system as the state variable $\boldsymbol{X}_{C M}$ (similar to that in reference [2], assuming that the target satellite signal carrier frequency $f_{0}$ is constant during observation), that is:

$$
\boldsymbol{X}_{C M}=\left[\begin{array}{ll}
\boldsymbol{r}^{\mathrm{T}} & \dot{\boldsymbol{r}}^{\mathrm{T}} f_{0} \tag{9.1}
\end{array}\right]^{\mathrm{T}}
$$

Therefore, the state differential equation of the satellite motion is as follows:

$$
\dot{\boldsymbol{X}}_{C M}=\boldsymbol{F}_{C M}\left(\boldsymbol{X}_{C M}\right)=\left[\begin{array}{lll}
\dot{\boldsymbol{r}}^{\mathrm{T}} & \ddot{\boldsymbol{r}}^{\mathrm{T}} & 0 \tag{9.2}
\end{array}\right]^{\mathrm{T}} .
$$

In the case of two-body model, according to the law of universal gravitation, the satellite motion equation is as follows [6]:

$$
\begin{equation*}
\ddot{r}=-\mu \frac{r}{r^{3}}, \tag{9.3}
\end{equation*}
$$

where $\boldsymbol{r}=\left[\begin{array}{lll}x & y & z\end{array}\right]^{\mathrm{T}}, \dot{\boldsymbol{r}}=\left[\begin{array}{lll}\dot{x} & \dot{y} & \dot{z}\end{array}\right]^{\mathrm{T}}$, and $\ddot{\boldsymbol{r}}=\left[\begin{array}{lll}\ddot{x} & \ddot{y} & \ddot{z}\end{array}\right]^{\mathrm{T}}$, respectively, are the position, velocity, and acceleration vectors of the target satellite in the J2000.0 coordinate system, $\boldsymbol{F}_{C M}(\cdot)$ is the nonlinear transformation of state variable $\boldsymbol{X}_{C M}, \mu$ is the constant of gravity, and $r=\sqrt{x^{2}+y^{2}+z^{2}}$ is the distance between the target satellite and the centroid of the earth.

### 9.2.1.2 Measurement Model

Assuming that the observing satellite could measure the azimuth angle $\beta_{k}$, the elevation angle $\varepsilon_{k}$, and the Doppler frequency $f_{k}$ of the target satellite by means of direction finding and frequency measurement, the definitions of them are as follows:

$$
\begin{align*}
& \beta_{k}=\arctan \left(\frac{\rho_{y}(k)}{\rho_{x}(k)}\right)+n_{\beta k} \quad\left(k=0,1, \ldots, N_{s}-1\right),  \tag{9.4}\\
& \varepsilon_{k}=\arctan \left(\frac{\rho_{z}(k)}{\sqrt{\rho_{x}^{2}(k)+\rho_{y}^{2}(k)}}\right)+n_{\varepsilon k} \quad\left(k=0,1, \ldots, N_{s}-1\right), \tag{9.5}
\end{align*}
$$

where $n_{\beta}$ and $n_{\varepsilon}$ are measurement noise at the azimuth angle and the elevation angle, and it is still assumed that the measurement noise is zero-mean Gaussian distributed and the variances
are $\sigma_{\beta}^{2}$ and $\sigma_{\varepsilon}^{2}$, respectively. Suppose $\rho=\left[\rho_{x}(k) \rho_{y}(k) \rho_{z}(k)\right]^{\mathrm{T}}$ is the position vector of the target satellite in the coordinate system of the centroid measurement station belonging to the observing satellite (see Section 2.3 for the definition of the coordinate system).
Since relative motion between the satellites results in a larger Doppler shift in receiving signal frequencies, the Doppler frequency $f_{k}$ of the received signal can be expressed as

$$
\begin{equation*}
f_{k}=f_{0}\left[1-\frac{\rho \cdot \dot{\rho}}{c \sqrt{\rho_{x}^{2}(k)+\rho_{y}^{2}(k)+\rho_{z}^{2}(k)}}\right]+n_{f k}\left(k=0,1, \ldots, N_{s}-1\right), \tag{9.6}
\end{equation*}
$$

where $c$ is the propagation speed of the electromagnetic wave in free space, $f_{0}$ is the unknown signal carrier frequency, and $n_{f}$ is the frequency measurement noise assumed to be the Gaussian white noise with zero mean and variance $\sigma_{f}^{2}$. Suppose $\dot{\rho}=\left[\begin{array}{lll}\dot{\rho}_{x} & \dot{\rho}_{y} & \dot{\rho}_{z}\end{array}\right]^{\mathrm{T}}$ contains three velocity components of the target satellite in the coordinate system of the centroid measurement station belonging to the observing satellite (ee Section 2.3 for definitions).
The measurement vector is defined as $\boldsymbol{Z}_{C M}(k)=\left[\beta_{k} \varepsilon_{k} f_{k}\right]^{\mathbf{T}}$ and the measurement vector can be expressed as a nonlinear function of state variable $\boldsymbol{X}_{C M}$ :

$$
\begin{equation*}
\boldsymbol{Z}_{C M}(k)=\boldsymbol{H}_{C M}\left(\boldsymbol{X}_{C M}(k)\right)+\boldsymbol{n}_{C M}(k), \tag{9.7}
\end{equation*}
$$

where $\boldsymbol{H}_{C M}(\cdot)$ is the nonlinear transformation from the measurement vector consisting of expressions (9.4) to (9.6) to the state variable $\boldsymbol{X}_{C M}$ and $\boldsymbol{n}_{C M}(k)=\left[n_{\beta}(k) n_{\varepsilon}(k) n_{f}(k)\right]^{\mathrm{T}}$ is the measured noise vector in the measuring azimuth angle, elevation angle, and Doppler frequency shift, whose corresponding covariance matrix can be expressed as

$$
E\left[\boldsymbol{n}_{C M}(k) \boldsymbol{n}_{C M}(k)^{T}\right]=\boldsymbol{R}_{C M}(k)=\left[\begin{array}{ccc}
\sigma_{\beta}^{2} & 0 & 0  \tag{9.8}\\
0 & \sigma_{\varepsilon}^{2} & 0 \\
0 & 0 & \sigma_{f}^{2}
\end{array}\right]
$$

### 9.2.2 Tracking Method in the Case of the Two-Body Model

Similar to the analytic method in Chapter 8, the state differential equation (Equation (9.2)) of the satellite can be discretized to obtain

$$
\begin{equation*}
\boldsymbol{X}_{C M}(k+1)-\boldsymbol{X}_{C M}(k)=\int_{t_{k}}^{t_{k+1}} \boldsymbol{F}_{C M}\left(\boldsymbol{X}_{C M}(t)\right) \mathrm{d} t \tag{9.9}
\end{equation*}
$$

When the time interval $t_{k+1}-t_{k}=T$ is short enough, $\boldsymbol{F}_{C M}\left(\boldsymbol{X}_{C M}(t)\right)$ can be expanded to a Taylor series in the vicinity of $t_{k}$ :

$$
\begin{equation*}
\boldsymbol{F}_{C M}\left(\boldsymbol{X}_{C M}(t)\right) \approx \boldsymbol{F}_{C M}\left(\boldsymbol{X}_{C M}(k)\right)+\boldsymbol{A}_{C M}\left(\boldsymbol{X}_{C M}(k)\right) \cdot \boldsymbol{F}_{C M}\left(\boldsymbol{X}_{C M}(k)\right)\left(t-t_{k}\right), \tag{9.10}
\end{equation*}
$$

where

$$
\boldsymbol{A}_{C M}\left(\boldsymbol{X}_{C M}(k)\right)=\left.\frac{\partial \boldsymbol{F}_{C M}\left(\boldsymbol{X}_{C M}(t)\right)}{\partial \boldsymbol{X}_{C M}}\right|_{t=t_{k}}
$$

is a $7 \times 7$ matrix. Substitute expression (9.10) into expression (9.9), which gives

$$
\begin{align*}
\boldsymbol{X}_{C M}(k+1)= & \boldsymbol{X}_{C M}(k)+\boldsymbol{F}_{C M}\left(\boldsymbol{X}_{C M}(k)\right) T+\boldsymbol{A}_{C M}\left(\boldsymbol{X}_{C M}(k)\right) \cdot \boldsymbol{F}_{C M}\left(\boldsymbol{X}_{C M}(k)\right) \frac{T^{2}}{2} \\
& +\boldsymbol{W}_{C M}(k) \tag{9.11}
\end{align*}
$$

where $\boldsymbol{W}_{C M}(k)$ is the error vector introduced when linearization is applied to the nonlinear state equation. It is of the same order as $\left|\boldsymbol{X}_{C M}(k)-\widehat{\boldsymbol{X}}_{C M}(k / k)\right|^{2}$ and $\mathrm{E}\left[\boldsymbol{W}_{C M}(k) \boldsymbol{W}_{C M}(k)^{\mathrm{T}}\right]=\boldsymbol{Q}_{C M}$ is the state noise covariance matrix.
According to expression (9.2) and the law of vector differentiation, $\boldsymbol{A}_{C M}\left(\boldsymbol{X}_{C M}(k)\right)$ can be expressed as follows:

$$
\boldsymbol{A}_{C M}\left(\boldsymbol{X}_{C M}(k)\right)=\left[\begin{array}{ccc}
\frac{\partial \dot{\boldsymbol{r}}}{\partial \boldsymbol{r}_{3 \times 3}} & \frac{\partial \dot{\boldsymbol{r}}}{\partial \dot{\boldsymbol{r}}_{3 \times 3}} & \frac{\partial \dot{\boldsymbol{r}}}{\partial f_{03 \times 1}}  \tag{9.12}\\
\frac{\partial \ddot{\boldsymbol{r}}}{\partial \boldsymbol{r}_{3 \times 3}} & \frac{\partial \dot{\boldsymbol{r}}}{\partial \dot{\boldsymbol{r}}_{3 \times 3}} & \frac{\partial \dot{\boldsymbol{r}}}{\partial f_{03 \times 1}} \\
\frac{\partial 0}{\partial \boldsymbol{r}_{1 \times 3}} & \frac{\partial 0}{\partial \dot{\boldsymbol{r}}_{1 \times 3}} & \frac{\partial 0}{\partial f_{01 \times 1}}
\end{array}\right]=\left[\begin{array}{cc}
\boldsymbol{A}(\boldsymbol{X}(k))_{6 \times 6} & \mathbf{0}_{6 \times 1} \\
\mathbf{0}_{1 \times 6} & 0
\end{array}\right]_{t=t_{k}}
$$

where $\boldsymbol{A}(\boldsymbol{X}(k))$ is the observation matrix of the bearings-only (BO) tracking method in the case of the two-body model (see Section 8.2.1) and $\boldsymbol{r}, \dot{\boldsymbol{r}}$, and $\ddot{\boldsymbol{r}}$ are respectively the position, velocity, and acceleration vector of the target satellite in the J2000.0 coordinate system.
By substituting the state filtered value $\widehat{\boldsymbol{X}}_{C M}(k / k)$ into expressions (9.12) and (9.11), respectively, the state prediction equation can be expressed as

$$
\begin{align*}
\hat{\boldsymbol{X}}_{C M}(k+1 / k)= & \hat{\boldsymbol{X}}_{C M}(k / k)+\boldsymbol{F}_{C M}\left(\hat{\boldsymbol{X}}_{C M}(k / k)\right) T \\
& +\boldsymbol{A}_{C M}\left(\hat{\boldsymbol{X}}_{C M}(k / k)\right) \cdot \boldsymbol{F}_{C M}(\widehat{\boldsymbol{X}}(k / k)) \frac{T^{2}}{2} . \tag{9.13}
\end{align*}
$$

The algorithm of the state transition matrix is now similar to that of BO in Chapter 8. Firstly, it can be expanded to a Taylor series in the vicinity of $t_{k}$ according to the definition [7] of the state transition matrix $\boldsymbol{\Phi}_{C M}\left(t, t_{k}\right)$ :

$$
\begin{equation*}
\boldsymbol{\Phi}_{C M}\left(t, t_{k}\right)=\boldsymbol{\Phi}_{C M}\left(t_{k}, t_{k}\right)+\left.\frac{\mathrm{d} \boldsymbol{\Phi}_{C M}\left(t, t_{k}\right)}{\mathrm{d} t}\right|_{t=t_{k}}\left(t-t_{k}\right)+O\left(t-t_{k}\right) . \tag{9.14}
\end{equation*}
$$

It can then be obtained after the continuous state transition matrix is discretized according to the property [7] of the state transition matrix that

$$
\begin{equation*}
\boldsymbol{\Phi}_{C M}(k+1 / k)=\boldsymbol{I}+\boldsymbol{A}_{C M}\left(\hat{\boldsymbol{X}}_{C M}(k / k)\right) T \tag{9.15}
\end{equation*}
$$

As the measurement equation is a nonlinear discrete equation, only linearization is required. The measurement equation can be expanded at $\widehat{\boldsymbol{X}}_{C M}(k+1 / k)$ to the Taylor series. As measurement values of both the angle and frequency are obtained in the body coordinate system of the observing satellite and the position and velocity components of the state variable are in the $\mathbf{J} 2000.0$ coordinate system, it is required to convert from the body coordinate system of the observing satellite to the J2000.0 coordinate system to calculate the Jacobian matrix. Based
on the law of vector differentiation [8], the measured Jacobian matrix can be expressed as

$$
\begin{align*}
\boldsymbol{H}_{C M}(k+1 / k) & =\left[\begin{array}{lll}
\frac{\partial \beta_{k}}{\partial \boldsymbol{r}_{1 \times 3}} & \frac{\partial \beta_{k}}{\partial \dot{r}_{1 \times 3}} & \frac{\partial \beta_{k}}{\partial f_{01 \times 1}} \\
\frac{\partial \varepsilon_{k}}{\partial \boldsymbol{r}_{1 \times 3}} & \frac{\partial \varepsilon_{k}}{\partial \dot{r}_{1 \times 3}} & \frac{\partial \varepsilon_{k}}{\partial f_{01 \times 1}} \\
\frac{\partial f_{k}}{\partial \boldsymbol{r}_{1 \times 3}} & \frac{\partial f_{k}}{\partial \dot{r}_{1 \times 3}} & \frac{\partial f_{k}}{\partial f_{01 \times 1}}
\end{array}\right] \\
& =\left[\begin{array}{lll}
\left(\frac{\partial \beta_{k}}{\partial \rho} \frac{\partial \rho}{\partial \boldsymbol{r}}+\frac{\partial \beta_{k}}{\partial \dot{\rho}} \frac{\partial \dot{\rho}}{\partial \boldsymbol{r}}\right)_{1 \times 3} & \frac{\partial \beta_{k}}{\partial \dot{\rho}} \frac{\partial \dot{\boldsymbol{\rho}}}{\partial \dot{\boldsymbol{r}}_{1 \times 3}} \frac{\partial \beta_{k}}{\partial f_{01 \times 1}} \\
\left(\frac{\partial \varepsilon_{k}}{\partial \boldsymbol{\rho}} \frac{\partial \rho}{\partial \boldsymbol{r}}+\frac{\partial \varepsilon_{k}}{\partial \dot{\rho}} \frac{\partial \dot{\rho}}{\partial \boldsymbol{r}}\right)_{1 \times 3} & \frac{\partial \varepsilon_{k}}{\partial \dot{\rho}} \frac{\partial \dot{\rho}}{\partial \dot{\boldsymbol{r}}_{1 \times 3}} \frac{\partial \varepsilon_{k}}{\partial f_{01 \times 1}} \\
\left(\frac{\partial f_{k}}{\partial \rho} \frac{\partial \rho}{\partial \boldsymbol{r}}+\frac{\partial f_{k}}{\partial \dot{\rho}} \frac{\partial \dot{\rho}}{\partial \boldsymbol{r}}\right)_{1 \times 3} & \frac{\partial f_{k}}{\partial \dot{\rho}} \frac{\partial \dot{\rho}}{\partial \dot{\boldsymbol{r}}_{1 \times 3}} \frac{\partial f_{k}}{\partial f_{01 \times 1}}
\end{array}\right] . \tag{9.16}
\end{align*}
$$

The following is the specific solution for the measured Jacobian matrix. According to expressions (9.4) and (9.5), $\boldsymbol{\beta}_{k}$ and $\varepsilon_{k}$ are the only functions of $\rho$, so it can be found that

$$
\begin{align*}
& \frac{\partial \beta_{k}}{\partial \dot{\rho}}=\frac{\partial \varepsilon_{k}}{\partial \dot{\rho}}=\mathbf{0}_{1 \times 3} \\
& \frac{\partial \beta_{k}}{\partial f_{0}}=\frac{\partial \varepsilon_{k}}{\partial f_{0}}=0 \tag{9.17}
\end{align*}
$$

According to the coordinate system definition in Section 2.3

$$
\begin{equation*}
\frac{\partial \rho}{\partial \boldsymbol{r}}=\boldsymbol{G}^{\mathrm{T}}, \frac{\partial \dot{\rho}}{\partial \boldsymbol{r}}=\dot{\boldsymbol{G}}^{\mathrm{T}}, \frac{\partial \dot{\boldsymbol{\rho}}}{\partial \dot{\boldsymbol{r}}}=\boldsymbol{G}^{\mathrm{T}} . \tag{9.18}
\end{equation*}
$$

According to the corresponding results of the BO tracking method in Chapter 8, the partial differential correlated with the angle measurement value can be expressed as

$$
\begin{align*}
& \frac{\partial \beta_{k}}{\partial \rho}=\left[\begin{array}{lll}
\frac{-\rho_{y}}{\rho_{x}^{2}+\rho_{y}^{2}} \frac{\rho_{x}}{\rho_{x}^{2}+\rho_{y}^{2}} & 0]_{1 \times 3} \\
\frac{\partial \varepsilon_{k}}{\partial \rho} & =\left[\frac{-\rho_{x} \rho_{z}}{\rho_{r}^{2} \sqrt{\rho_{x}^{2}+\rho_{y}^{2}}} \frac{-\rho_{y} \rho_{z}}{\rho_{r}^{2} \sqrt{\rho_{x}^{2}+\rho_{y}^{2}}} \frac{\sqrt{\rho_{x}^{2}+\rho_{y}^{2}}}{\rho_{r}^{2}}\right]_{1 \times 3} .
\end{array} . .\right. \tag{9.19}
\end{align*}
$$

The following is the solution for the partial differential correlated with the frequency measurement. It can be obtained according to expression (9.6) as follows:

$$
\begin{align*}
& \frac{\partial f_{k}}{\partial \rho}=\frac{f_{0}}{c}\left(\frac{(\rho \cdot \dot{\rho}) \rho-\rho_{r}^{2} \dot{\rho}}{\rho_{r}^{3}}\right) \\
& \frac{\partial f_{k}}{\partial \dot{\rho}}=-\frac{f_{0}}{c}\left(\frac{\rho}{\rho_{r}}\right) \frac{\partial f_{k}}{\partial f_{0}}=1-\frac{(\rho \cdot \dot{\rho})}{c \rho_{r}} \tag{9.21}
\end{align*}
$$

where the definitions of $\boldsymbol{\rho}, \dot{\boldsymbol{\rho}}, \boldsymbol{G}$, and $\dot{\boldsymbol{G}}$ are as shown in Section 2.3, and $\rho_{r}=\sqrt{\rho_{x}^{2}+\rho_{y}^{2}+\rho_{z}^{2}}$ is the distance between the observing satellite and the target satellite. Substituting expressions (9.17) to (9.21) into expression (9), the measured Jacobian matrix can be derived.

After being linearized and discretized, the state equation and the measurement equation can be substituted into the EKF expression for iterative calculation according to the following steps:

1. Calculate the one-step state prediction estimation $\widehat{\boldsymbol{X}}_{C M}(k+1 / k)$ as expression (9) and thencalculate its corresponding covariance matrix:

$$
\begin{equation*}
\boldsymbol{P}_{C M}(k+1 / k)=\boldsymbol{\Phi}_{C M}(k+1 / k) \boldsymbol{P}_{C M}(k / k) \boldsymbol{\Phi}_{C M}^{T}(k+1 / k)+\boldsymbol{Q}_{C M} . \tag{9.22}
\end{equation*}
$$

Here the constant matrix [7] can be selected for the $\boldsymbol{Q}_{C M}$ matrix by experience.
2. Calculate the filter gain matrix:

$$
\begin{align*}
\boldsymbol{K}_{C M}(k+1)= & \boldsymbol{P}_{C M}(k+1 / k) \boldsymbol{H}_{C M}^{T}(k+1 / k) \\
& \left(\boldsymbol{H}_{C M}(k+1 / k) \boldsymbol{P}_{C M}(k / k) \boldsymbol{H}_{C M}^{T}(k+1 / k)+\boldsymbol{R}_{C M}(k+1)\right)^{-1} . \tag{9.23}
\end{align*}
$$

3. Calculate the state filter update and the corresponding covariance matrix:

$$
\begin{align*}
\hat{\boldsymbol{X}}_{C M}(k+1 / k+1)= & \hat{\boldsymbol{X}}_{C M}(k+1 / k)+\boldsymbol{K}_{C M}(k+1) \\
& \left(\boldsymbol{Z}_{C M}(k+1)-\boldsymbol{H}_{C M}\left(\hat{\boldsymbol{X}}_{C M}(k+1 / k)\right)\right),  \tag{9.24}\\
\boldsymbol{P}_{C M}(k+1)= & \left(\boldsymbol{I}-\boldsymbol{K}_{C M}(k+1) \boldsymbol{H}_{C M}(k+1 / k)\right) \boldsymbol{P}_{C M}(k+1 / k) . \tag{9.25}
\end{align*}
$$

### 9.2.3 Mathematical Models Considering $J_{2}$ Perturbation of Earth Oblateness

Since it is idealistic to analyze satellite-to-satellite passive tracking performance in the case of the two-body model, this section will consider the influence of orbit perturbation of earth oblateness.
The state variable considering perturbation is the same as the two-body model and is still $\boldsymbol{X}_{C M}$, as shown in expression (9.1). According to expression (9.1), the state differential equation of satellite motion when the term $J_{2}$ perturbation of earth oblateness is considered is as follows:

$$
\dot{\boldsymbol{X}}_{C M}=\overline{\boldsymbol{F}}_{C M}\left(\boldsymbol{X}_{C M}\right)=\left[\begin{array}{lll}
\dot{\boldsymbol{r}}^{\mathrm{T}} & \ddot{\boldsymbol{r}}^{\mathrm{T}} & 0 \tag{9.26}
\end{array}\right]^{\mathrm{T}} .
$$

In order to distinguish these results from the results in the case of the two-body model, $\overline{\boldsymbol{F}}_{C M}(\cdot)$ is adopted as the nonlinear transformation against the state variable $\boldsymbol{X}_{C M}$. The key to solve expression (9.26) is to calculate the perturbation acceleration of the satellite at this moment.

The perturbation acceleration is derived in Chapter 8, which is expressed as:

$$
\left.\begin{array}{l}
\ddot{x}=-\frac{\mu}{r^{3}} x\left[1-J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(7.5 \frac{z^{2}}{r^{2}}-1.5\right)\right] \\
\ddot{y}=-\frac{\mu}{r^{3}} y\left[1-J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(7.5 \frac{z^{2}}{r^{2}}-1.5\right)\right]  \tag{9.27}\\
\ddot{z}=-\frac{\mu}{r^{3}} z\left[1-J_{2}\left(\frac{R_{e}}{r}\right)^{2}\left(7.5 \frac{z^{2}}{r^{2}}-4.5\right)\right]
\end{array}\right\},
$$

where $R_{e}$ is the average equatorial radius of the earth and $J_{2}$ is the second-order spherical harmonic coefficient of terrestrial gravitation field.
Whether perturbation is being considered has nothing to do with the observations as the measurement model considering orbit perturbation of the earth oblateness is the same as that in Section 9.2 .1 in the case of the two-body model.

### 9.2.4 Tracking Method Considering $\boldsymbol{J}_{2}$ Perturbation of Earth Oblateness

Considering $\mathrm{J}_{2}$ perturbation of earth oblateness, the state differential equation of satellite motion is different from that in the case of the two-body model, so the corresponding state prediction equation and state transition matrix are also different from those in the case of the two-body model, while the measured Jacobian matrix is the same as that of the two-body model. The following is only to derive the difference from the results in the case of the two-body model.
Similarly, the state differential equation (Equation (9.26)) of the target satellite is now discretized to obtain

$$
\begin{equation*}
\boldsymbol{X}_{C M}(k+1)-\boldsymbol{X}_{C M}(k)=\int_{t_{k}}^{t_{k+1}} \overline{\boldsymbol{F}}_{C M}\left(\boldsymbol{X}_{C M}(t)\right) \mathrm{d} t \tag{9.28}
\end{equation*}
$$

When the time interval $t_{k+1}-t_{k}=T$ is short enough, $\overline{\boldsymbol{F}}_{C M}\left(\boldsymbol{X}_{C M}(t)\right)$ can be expanded to a Taylor series in the vicinity of $t_{k}$ :

$$
\begin{equation*}
\overline{\boldsymbol{F}}_{C M}\left(\boldsymbol{X}_{C M}(t)\right) \approx \overline{\boldsymbol{F}}_{C M}\left(\boldsymbol{X}_{C M}(k)\right)+\overline{\boldsymbol{A}}_{C M}\left(\boldsymbol{X}_{C M}(k)\right) \cdot \overline{\boldsymbol{F}}_{C M}\left(\boldsymbol{X}_{C M}(k)\right)\left(t-t_{k}\right) \tag{9.29}
\end{equation*}
$$

Where

$$
\overline{\boldsymbol{A}}_{C M}\left(\boldsymbol{X}_{C M}(k)\right)=\left.\frac{\partial \overline{\boldsymbol{F}}_{C M}\left(\boldsymbol{X}_{C M}(t)\right)}{\partial \boldsymbol{X}_{C M}}\right|_{t=t_{k}}
$$

is a $7 \times 7$ matrix. Substituting expression (9.29) into expression (9.28) yields

$$
\begin{align*}
\boldsymbol{X}_{C M}(k+1)= & \boldsymbol{X}_{C M}(k)+\overline{\boldsymbol{F}}_{C M}\left(\boldsymbol{X}_{C M}(k)\right) T+\overline{\boldsymbol{A}}_{C M}\left(\boldsymbol{X}_{C M}(k)\right) \cdot \overline{\boldsymbol{F}}_{C M}\left(\boldsymbol{X}_{C M}(k)\right) \frac{T^{2}}{2} \\
& +\overline{\boldsymbol{W}}_{C M}(k) . \tag{9.30}
\end{align*}
$$

In the above expression, $\bar{W}_{C M}(k)$ is still the error vector introduced when the nonlinear state equation is linearized, while it is different from that of the two-body model $\boldsymbol{W}_{C M}(k)$. To make a distinction, here it is $\overline{\boldsymbol{W}}_{C M}(k)$, which is still on the same order as $\left|\boldsymbol{X}_{C M}(k)-\widehat{\boldsymbol{X}}_{C M}(k / k)\right|^{2}$ and $\mathrm{E}\left[\overline{\boldsymbol{W}}_{C M}(k) \overline{\boldsymbol{W}}_{C M}(k)^{\mathrm{T}}\right]=\overline{\boldsymbol{Q}}_{C M}$ and is still the state noise covariance matrix.
According to expression (9.26) and the vector differential law, $\overline{\boldsymbol{A}}_{C M}\left(\boldsymbol{X}_{C M}(k)\right)$ can be expressed as follows:

$$
\overline{\boldsymbol{A}}_{C M}\left(\boldsymbol{X}_{C M}(k)\right)=\left[\begin{array}{ccc}
\frac{\partial \dot{\boldsymbol{r}}}{\partial \boldsymbol{r}_{3 \times 3}} & \frac{\partial \dot{\boldsymbol{r}}}{\partial \dot{\boldsymbol{r}}_{3 \times 3}} & \frac{\partial \dot{\boldsymbol{r}}}{\partial f_{03 \times 1}}  \tag{9.31}\\
\frac{\partial \ddot{\boldsymbol{r}}}{\partial \boldsymbol{r}_{3 \times 3}} & \frac{\partial \dot{\boldsymbol{r}}}{\partial \dot{\boldsymbol{r}}_{3 \times 3}} & \frac{\partial \dot{\boldsymbol{r}}}{\partial f_{03 \times 1}} \\
\frac{\partial 0}{\partial \boldsymbol{r}_{1 \times 3}} & \frac{\partial 0}{\partial \dot{\boldsymbol{r}}_{1 \times 3}} & \frac{\partial 0}{\partial f_{01 \times 1}}
\end{array}\right]=\left[\begin{array}{cc}
\overline{\boldsymbol{A}}(\boldsymbol{X}(k))_{6 \times 6} & \mathbf{0}_{6 \times 1} \\
\mathbf{0}_{1 \times 6} & 0
\end{array}\right]_{t=t_{k}}
$$

where $\overline{\boldsymbol{A}}(\boldsymbol{X}(k))$ is the derived matrix of the BO tracking method considering $J_{2}$ perturbation in Chapter 8 and $\boldsymbol{r}, \dot{\boldsymbol{r}}$, and $\ddot{\boldsymbol{r}}$ are respectively the position, velocity, and acceleration vector of the target satellite in the J 2000.0 coordinate system.
Substitute the state filtered value $\widehat{\boldsymbol{X}}_{C M}(k / k)$ into expressions (9.31) and (9.30) respectively. The state prediction equation considering $J_{2}$ perturbation of the earth oblateness can then be expressed as

$$
\begin{align*}
\hat{\boldsymbol{X}}_{C M}(k+1 / k)= & \widehat{\boldsymbol{X}}_{C M}(k / k)+\overline{\boldsymbol{F}}_{C M}\left(\hat{\boldsymbol{X}}_{C M}(k / k)\right) T \\
& +\overline{\boldsymbol{A}}_{C M}\left(\widehat{\boldsymbol{X}}_{C M}(k / k)\right) \cdot \overline{\boldsymbol{F}}_{C M}(\widehat{\boldsymbol{X}}(k / k)) \frac{T^{2}}{2} . \tag{9.32}
\end{align*}
$$

Calculate the state transition matrix by a method similar to that in the case of the two-body model. Firstly, obtain the continuous state transition matrix. Then expand this to a Taylor series in the vicinity of $t_{k}$. Finally, discretize it. The state transition matrix after being discretized is found as $\overline{\boldsymbol{\Phi}}_{C M}(k+1 / k)$ and its expression is

$$
\begin{equation*}
\overline{\boldsymbol{\Phi}}_{C M}(k+1 / k)=\boldsymbol{I}+\overline{\boldsymbol{A}}_{C M}\left(\widehat{\boldsymbol{X}}_{C M}(k / k)\right) T . \tag{9.33}
\end{equation*}
$$

### 9.3 System Observability Analysis

Chapter 8 of this book has given a detailed analysis of the existing observability problem and calculation methods. It introduced a kind of system observability description method and analyzed the advantages of using the method to describe the system observability compared with the existing description method. This section will analyze the observability of the satellite-to-satellite passive tracking system based on angle and frequency information using the description method. Since the known carrier frequency $f_{0}$ is a special case of the unknown $f_{0}$ in Section 9.2 , this section will only analyze system observability of targeting the unknown $f_{0}$ situations.
Still taking the typical simulation scenario in Chapter 8 as an example, we shall study the relations between the system observability and correlated influencing factors of the state
equation and the measurement equation by simulation. Suppose the filtering period is $T$, the observation duration is 5000 seconds, the angle measurement errors are $\sigma_{\beta}$ and $\sigma_{\varepsilon}$, with $\sigma_{\beta}=\sigma_{\varepsilon}$, the frequency measurement error is $\sigma_{f}$, and the signal carrier frequency of the target satellite is $f_{0}$.

### 9.3.1 Influence of Factors of the State Equation

When the carrier frequency $f_{0}$ is unknown, the state equation of the satellite-to-satellite passive tracking system based on angle and frequency information can be expressed as Equation (9.2). According to expression (9.15), the state transition matrix after being discretized is expressed as

$$
\boldsymbol{\Phi}_{C M}(k+1 / k)=\boldsymbol{I}+\boldsymbol{A}_{C M}\left(\hat{\boldsymbol{X}}_{C M}(k / k)\right) T
$$

As $\boldsymbol{\Phi}_{C M}(k+1 / k)$ is basically determined by the filtering period $T$ and changes slightly with variation of the satellite position parameter, analyses on relations between correlated influencing factors of the state equation and system observability focus on studying the influence of the filtering period $T$ on system observability.
Assume that the angle measurement error is $\sigma_{\beta}=20^{\prime \prime}$, the frequency measurement error is $\sigma_{f}=1 \mathrm{kHz}$, and the signal carrier frequency of the target satellite is $f_{0}=15 \mathrm{GHz}$. Select filtering periods as $T=5$ seconds, $T=20$ seconds, and $T=50$ seconds, respectively, to test the variation curves of the position and the velocity estimation errors and the system observability calculated in expression (8.42). At the same time, a comparison of the condition numbers of the system observability method [9] and the BO method, and the observability curve of the position and the velocity error curves are shown in Figure 9.1.
According to Figure 9.1, the shorter the filtering period, the smaller is the system observability value and the better the system observability, the higher is the corresponding position and velocity estimation precision. Conversely, the longer the filtering period, the worse is the system observability and the lower is the corresponding position and velocity estimation precision. In other words, the filtering period is an important parameter based on the passive tracking system and combines the bearings and frequency methods, which directly determines the state transition matrix of the system and consequently influences the system observability.
Moreover, according to the corresponding results of Figure 9.1a and b and the BO method in Chapter 8, when the condition number is now adopted to describe the system observability, the obtained observability of the CM is worse than that of BO method but the truth is not the same, which indicates that using the condition number to describe observability does not apply here. Consequently, when the observability description method introduced in Chapter 8 is adopted, observability obtained from the CM is obviously superior to that of the BO method.

### 9.3.2 Influence of Factors of the Measurement Equation

When the carrier frequency $f_{0}$ is unknown, the measurement equation of the satellite-to-satellite passive tracking system based on the angle and frequency is expression (9.7) and the corresponding measurement Jacobian matrix is expression (9). Here the research will be focused on the influence of the angle measurement error $\sigma_{\beta}$, the frequency measurement error $\sigma_{f}$, and the signal carrier frequency $f_{0}$ of target satellite on the system observability when
analyzing relations between the system observability and correlated influencing factors of the measurement equation.

### 9.3.2.1 Relations between the System Observability and the Angle Measurement Error

Now assume that the filtering period is $T=5$ seconds, the frequency measurement error is $\sigma_{f}=100 \mathrm{~Hz}$, the signal carrier frequency of the target satellite is $f_{0}=15 \mathrm{GHz}$, and the select angle measurement errors $\sigma_{\beta}=10^{\prime \prime}, \sigma_{\beta}=50^{\prime \prime}$, and $\sigma_{\beta}=150^{\prime \prime}$, respectively, are used to test the variation curve of the position and velocity estimation errors and the system observability introduced in this section under different angle measurement errors. Similarly, the condition number method for the measurement of the system observability is determined [9] and the BO


Figure 9.1 Relations between system observability and location and velocity estimation error and filtering period. (a) Observability curve indicated with condition number, (b) observability calculated according to expression (8.42), (c) position estimation error curve, and (d) velocity estimation error curve


Figure 9.1 (continued)
method is used as a comparison, and the observability curve and position and velocity error curves are shown in Figure 9.2.
According to Figure 9.2, the shorter the angle measurement error, the smaller is the system observability value and the better the system observability, the higher is the corresponding position and velocity estimation precision. Conversely, the greater the angle measurement error, the worse is the system observability and the lower is the corresponding position and velocity estimation precision. In other words, for the satellite-to-satellite passive tracking system with the CM, the angle measurement error is also an important factor that influences the system observability.
Moreover, according to Figure 9.2a to d, the system observability describing the condition number can not only show the influence of angle measurement error on the system observability but also draws the conclusion that the system observability based on the CM is worse than that of the BO method, which is inconsistent with the simulation results in Figure 9.2e and f. Use of the system observability description method introduced in this section can clearly show the relations between the observability and angle measurement error. Along with the increase of the angle measurement error, the system observability value is obviously increased, and
the system observability obtained by the CM is apparently superior to that of the BO method under the same conditions.

### 9.3.2.2 Relations between the System Observability and the Frequency Measurement Error

Assume that the filtering period is still $T=5$ seconds, the angle measurement error is $\sigma_{\beta}=20^{\prime \prime}$, and the signal carrier frequency of the target satellite is $f_{0}=15 \mathrm{GHz}$. Select the angle measurement errors $\sigma_{f}=100 \mathrm{~Hz}, \sigma_{f}=5 \mathrm{kHz}$, and $\sigma_{f}=50 \mathrm{kHz}$, respectively, to test the variation curve of the position, the velocity estimation error, and the system observability introduced in this section under different frequency measurement errors. Similarly, the condition number method as the measurement of the system observability [9] is used, with the BO method used


Figure 9.2 Relations between system observability and location and velocity estimation error and angle measurement. (a) Observability curve indicated with condition number, (b) observability calculated according to expression (8.42) when $\sigma_{\beta}=10^{\prime \prime}$, (c) observability calculated according to expression (8.42) when $\sigma_{\beta}=50^{\prime \prime}$, (d) observability calculated according to expression (8.42) when $\sigma_{\beta}=150^{\prime \prime}$, (e) position estimation error curve, and (f) velocity estimation error curve


Figure 9.2 (continued)
as a comparison; the observability curve and the position and velocity error curves are shown in Figure 9.3.
According to Figure 9.3, the shorter the frequency measurement error, the smaller is the system observability value and the better the system observability, the higher is the corresponding position and velocity estimation precision. Conversely, the greater the frequency measurement error, the worse is the system observability and the lower is the corresponding position and velocity estimation precision. In other words, for the satellite-to-satellite passive tracking system with the CM, the frequency measurement error is also an important factor that influences the system observability.
According to Figure 9.3a and b , the system observability describing the condition number still cannot reflect the influence of the frequency measurement error, but the observability description method introduced in this section can clearly show the influence of the frequency measurement error. Along with the increase of the frequency measurement error, the system observability value is also increased, and the observability obtained through the CM is obviously superior to that of the BO method. Moreover, according to Figure 9.3b, relations between the system observability and the frequency measurement error are not uniformly decreased,
and with the increase of the frequency measurement error, the system observability is gradually decreased, which is reflected in Figure 9.3c and d.

### 9.3.2.3 Relations between the System Observability and the Target Signal Carrier Frequency $f_{0}$

Assume that the filtering period is still $T=5$ seconds, the angle measurement error is $\sigma_{\beta}=20^{\prime \prime}$, and the relative frequency error is $\sigma f / f_{0}=5 \times 10^{-8}$. Select the signal carrier frequencies of the target satellite $f_{0}=1 \mathrm{GHz}, f_{0}=3 \mathrm{GHz}$, and $f_{0}=20 \mathrm{GHz}$, respectively, to test the variation curve of the position and velocity estimation errors, and the system observability introduced in this section under the same relative frequency measurement error. Similarly, the condition number


Figure 9.3 Relations between system observability and location and velocity estimation error and frequency measurement. (a) Observability curve indicated with condition number, (b) observability calculated according to expression (8.42), (c) position estimation error curve, and (d) velocity estimation error curve


Figure 9.3 (continued)
method as the measurement of the system observability [9] is used, with the BO method used for comparison, and the observability curve and position and velocity error curves are shown in Figure 9.4.

According to Figure 9.4, the system observability curves obtained under the same relative frequency measurement error $\sigma f / f_{0}$ and the different signal carrier frequencies of the target satellite $f_{0}$ are the same, indicating that the system observability has nothing to do with the target signal carrier frequency $f_{0}$ for the satellite-to-satellite passive tracking system using the CM. This compares well with the position and velocity estimation error curves in Figure 9.4.
According to Figure 9.4a and b, the system observability curves describing the condition number is different when $f_{0}=20 \mathrm{GHz}, f_{0}=1 \mathrm{GHz}$, and $f_{0}=3 \mathrm{GHz}$. Curves obtained through the observability description method introduced in this section arestill the same when $f_{0}$ is given different values, from which the clear conclusion can be drawn that the system observability using the CM only relates to the relative frequency measurement error $\sigma f / f_{0}$ and has nothing to do with $f_{0}$.

In conclusion, the simulation analysis is applied to various factors influencing the satellite-to-satellite passive tracking system based on the CM. The conclusions are as follows:

1. The method [9] using the condition number as the system observability is not applicable to satellite-to-satellite passive orbit determination application. Firstly, it cannot reflect the relations between the observability and measurement errors; secondly, it cannot correctly reflect the situation that the system observability using the CM is better than that of the BO method; and thirdly, the observability curves obtained under the same relative frequency measurement error $\sigma f / f_{0}$ still relate to $f_{0}$. The observability description method introduced in this section could well solve the above-mentioned problem.
2. For the satellite-to-satellite passive tracking system based on the CM, the filtering period has the most significant effect on the system observability, the angle measurement error comes second, and the relative frequency measurement error comes third, while the target signal carrier $f_{0}$ has nothing to do with it. Therefore, it can offer a certain reference frame to the satellite-to-satellite passive tracking performance estimation based on the CM.


Figure 9.4 Relations between system observability and location and velocity estimation error and signal carrier frequency of the target satellite $f_{0}$. (a) Observability curve indicated with condition number, (b) observability calculated according to expression (8.42), (c) position estimation error curve, and (d) velocity estimation error curve


Figure 9.4 (continued)

### 9.4 Simulation and Its Analysis

In this section, the simulation of performances for the CM based on satellite-to-satellite passive orbit determination and the tracking system discussed in Sections 9.2 and 9.3 will be analyzed under different kinds of simulation conditions. The system models will still be considered in two conditions, that is, in the two-body model and in considering $J_{2}$ perturbation of earth oblateness. To evaluate the performances of the passive tracking system, here the CRLB (Cramér-Rao lower bound) under corresponding conditions is also introduced to the stimulation results to test whether the algorithm is the optimal and how far it can be improved.
Factors influencing the performances of the nonlinear filter usually include the measurement error, the initial state error, and the target model error, and so on. According to the analysis results of the system observability in Section 9.3, this section will simulate the performances of the CM based on satellite-to-satellite passive tracking under the initial state error, the angle measurement error, the frequency measurement error, the filter period and the ephemeris error of the observer itself, will analyze the effects of different factors on tracking accuracy and, finally, will arrive at corresponding conclusions.

Incomparison with the BO tracking method, typical simulation scenarios used in Chapter 2 will be adopted here. Also, if the algorithm in this chapter can converge within the shortest observable time span, the shielding effect of the earth may not be considered when the observing satellite passively determines the orbit of the target satellite and tracks it. Because the known carrier frequency $f_{0}$ is a special case of the unknown $f_{0}$, the CM in the following simulations refers to the unknown signal carrier frequency $f_{0}$ unless otherwise specified.

### 9.4.1 Simulation in the Case of the Two-Body Model

### 9.4.1.1 Influences of Different Initial State Errors on Passive Tracking Performance

Assume that the angle measurement error is $\sigma_{\beta}=20^{\prime \prime}$, the filter period is $T=2$ seconds, the frequency measurement error is $\sigma_{f}=100 \mathrm{~Hz}$, the target satellite signal carrier frequency is $f_{0}=15 \mathrm{GHz}$, and define three different groups of initial state error $\boldsymbol{X}_{\rho^{C M}}^{C M}$, which are $\left[\begin{array}{lllllll}100 & 100 & 100 & 0.1 & 0.1 & 0.1 & 5 e 4\end{array}\right]^{\mathrm{T}}$ (Case 2), $\left[\begin{array}{lllllll}300 & 300 & 300 & 0.5 & 0.5 & 0.5 & 1 e 5\end{array}\right]^{\mathrm{T}}$ (Case 3), and $\left[\begin{array}{lllllll}500 & 500 & 500 & 1 & 1 & 1 & 1 \\ e\end{array}\right]^{\mathrm{T}}$ (Case 4), respectively, among which the units of the initial state errors are: position vector $\boldsymbol{r}$ in kilometers, velocity vector $\dot{\boldsymbol{r}}$ in kilometers per second, and target signal carrier frequency $f_{0}$ in hertz. In addition, the method suggested in reference [10] are also introduced here for comparison, and its initial state error is $\left[\begin{array}{lllll}200 & 0.0005 & 3 & 3 & 3\end{array} 0\right]^{\mathrm{T}}$ (Case 1). Since the CRLBs for different initial state errors are identical, here the CRLB of the CM is introduced as the reference for the CRLB. According to the estimated location and velocity vectors of each point, the position and velocity estimation errors of the target satellite can be calculated as shown in Figure 9.5.
For these groups of simulation conditions, the corresponding estimation error and corresponding CRLB of the converged position, velocity, and target signal carrier frequency $f_{0}$ are shown respectively in Table 9.1.
According to Figure 9.5 and Table 9.1 it can be seen that the CM in the case of the two-body model in this chapter can well adapt to different initial state errors and steadily converge at last. With the increase of the initial state error, both the filtering convergence time and the final converged error will slightly increase; when the position error increases by about 30 m and the velocity error increases by about $0.1 \mathrm{~m} / \mathrm{s}$, the performance of the algorithm is very close to the CRLB. However, if the method suggested in reference [10] is adopted, since the circular orbit target satellite will be confronted with the singularity problem when classic orbital elements are adopted as the state variable, its filter performance is greatly influenced, and both the filtering convergence time and the estimation error greatly increase in comparison with the CM.

### 9.4.1.2 Influences of Different Angle Measurement Errors on Passive Tracking Performance

Assume that the filter period is $T=2$ seconds, the initial state error is $\left[\begin{array}{llll}300 & 300 & 300 & 0.5 \\ 0.5\end{array}\right.$ $0.51 e 5]^{\mathrm{T}}$ (the unit is the same as that in Simulation 1), the frequency measurement error is $\sigma_{f}=100 \mathrm{~Hz}$, the target satellite signal carrier frequency is $f_{0}=15 \mathrm{GHz}$, and the passive tracking estimation error can be calculated for the angle measurement error $\sigma_{\beta}$ when $10^{\prime \prime}, 50^{\prime \prime}, 150^{\prime \prime}$, and $200^{\prime \prime}$, respectively. For easy comparison, the BO method and the CRLB in Chapter 8 are here introduced as references. Since the CRLB of the CM is different for different angle


Figure 9.5 Position and velocity error curves of different initial state errors in the case of the two-body model. (a) Position estimation error curve and (b) velocity estimation error curve

Table 9.1 Parameter estimation error of different initial state errors in the case of the two-body model

| Simulation <br> scenarios/parameters | Case 1 | Case 2 | Case 3 | Case 4 | CRLB |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Position estimation error <br> $(\mathrm{km})$ | 31.7958 | 0.3192 | 0.3307 | 0.3677 | 0.3297 |
| Velocity estimation error <br> $(\mathrm{km} / \mathrm{s})$ | 0.0155 | $3.4647 \times 10^{-4}$ | $3.8354 \times 10^{-4}$ | $3.8432 \times 10^{-4}$ | $3.5255 \times 10^{-4}$ |
| Target carrier frequency <br> $\quad$ error $(\mathrm{Hz})$ | - | 3.1 | 3.9 | 7.8 | 7.2 |



Figure 9.6 Position and velocity error curve in the case of the two-body model when the angle measurement error is $\sigma_{\beta}=10^{\prime \prime}$. (a) Position estimation error curve and (b) velocity estimation error curve
measurement errors, the estimation error curves for different angle measurement errors are here given respectively:

1. When the angle measurement error is $\sigma_{\beta}=10^{\prime \prime}$, the estimation results are shown in Figure 9.6.
2. When the angle measurement error is $\sigma_{\beta}=50^{\prime \prime}$, the estimation results are shown in Figure 9.7.
3. When the angle measurement error is $\sigma_{\beta}=150^{\prime \prime}$, the estimation results are shown in Figure 9.8.
4. When the angle measurement error is $\sigma_{\beta}=200^{\prime \prime}$, the estimation results are shown in Figure 9.9.

In the above four groups of simulation conditions, the corresponding converged position, velocity, target signal carrier frequency $f_{0}$ estimation error, and corresponding CRLB are shown respectively in Table 9.2.

According to Figures 9.6 to 9.9 and Table 9.2, for the two-body model under different angle measurement errors, the accuracy of orbit determination and tracking of the CM is obviously better than the BO method, and its advantage becomes more and more obvious with the increase of angle measurement errors, and both the filtering convergence time,


Figure 9.7 Position and velocity error curves in the case of the two-body model when the angle measurement error is $\sigma_{\beta}=50^{\prime \prime}$. (a) Position estimation error curve and (b) velocity estimation error curve
position accuracy, and velocity accuracy are clearly improved. In addition, when the angle measurement error is less than $200^{\prime \prime}$, the position and velocity estimation error curves of the CM are close to the corresponding CRLB curves, indicating that the tracking accuracy of the CM in the case of the two-body model is close to the optimal estimation performance.

With other simulation conditions fixed, now test the influence of different angle measurement errors on the performance of passive tracking used when adopting the CM and $f_{0}$ is known. In order to compare the performances, corresponding estimation results of unknown $f_{0}$ and CRLB are here introduced as references (the CRLB at this point of time is the lower estimation error bound when $f_{0}$ is known).


Figure 9.8 Position and velocity error curves in the case of the two-body model when the angle measurement error is $\sigma_{\beta}=150^{\prime \prime}$. (a) Position estimation error curve and (b) velocity estimation error curve


Figure 9.9 Position and velocity error curves in the case of the two-body model when the angle measurement error is $\sigma_{\beta}=200^{\prime \prime}$. (a) Position estimation error curve and (b) velocity estimation error curve
5. When $f_{0}$ is known and the angle measurement error is $\sigma_{\beta}=10^{\prime \prime}$, the estimation result is shown in Figure 9.10.
6. When $f_{0}$ is known and the angle measurement error is $\sigma_{\beta}=150^{\prime \prime}$, the estimation result is shown in Figure 9.11.

In the above two groups of simulation conditions, the corresponding converged position, velocity estimation error, and the CRLB are shown respectively in Table 9.3.

According to Figure 9.10, Figure 9.11, and Table 9.3, for the different angle measurement errors of the two-body model, when the CM is adopted, the passive tracking accuracy of known $f_{0}$ has been improved in comparison with unknown $f_{0}$, but as the angle measurement errors increase, the filtering convergence time of the known $f_{0}$ case also greatly increases, and its advantage relative to unknown $f_{0}$ is only reflected in position estimation accuracy.
Table 9.2 Parameter estimation error of different angle measurement errors in the case of the two-body model

| Simulation scenarios/Parameters | $\sigma_{\beta}=10^{\prime \prime}$ |  |  | $\sigma_{\beta}=50^{\prime \prime}$ |  |  | $\sigma_{\beta}=150{ }^{\prime \prime}$ |  |  | $\sigma_{\beta}=200^{\prime \prime}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | BO <br> location error | CM <br> location error | CRLB | BO <br> location error | CM <br> location error | CRLB | BO <br> locationl error | CM <br> location error | CRLB | BO location error | CM <br> location error | CRLB |
| Position estimation error (km) | 0.9114 | 0.2105 | 0.2089 | 1.8138 | 1.2450 | 0.7026 | 4.2059 | 3.2094 | 1.8198 | 9.5796 | 4.8301 | 2.2971 |
| Velocity estimation error (km/s) | $9.5498 \times 10^{-4}$ | $2.2377 \times 10^{-4}$ | $2.0978 \times 10^{-4}$ | 0.0020 | 0.0017 | $8.8315 \times 10^{-4}$ | ${ }^{4} 0.0050$ | 0.0049 | 0.0022 | 0.0105 | 0.0074 | 0.0028 |
| Target carrier frequency error (Hz) | - | 2.3 | 6.3 | - | 5.5 | 11.4 | - | 33.0 | 21.9 | - | 53.2 | 27.1 |



Figure 9.10 Position and velocity error curves in the case of the two-body model when $f_{0}$ is known and the angle measurement error is $\sigma_{\beta}=10^{\prime \prime}$. (a) Position estimation error curve and (b) velocity estimation error curve

### 9.4.1.3 Influences of Different Frequency Measurement Errors on Passive Tracking Performances

Assume that the filter period is $T=2$ seconds, the initial state error is still $\left[\begin{array}{lll}300 & 300 & 300 \\ 0.5\end{array}\right.$ $0.50 .51 e 5]^{\mathrm{T}}$, the angle measurement error is $\sigma_{\beta}=50^{\prime \prime}$, and the target signal carrier frequency is $f_{0}=15 \mathrm{GHz}$, We can calculate the passive tracking estimation errors for frequency measurement errors being $\sigma_{f}=50 \mathrm{~Hz}$ (Case 1), $\sigma_{f}=500 \mathrm{~Hz}$ (Case 2), $\sigma_{f}=5 \mathrm{kHz}$ (Case 3), and $\sigma_{f}=50 \mathrm{kHz}$ (Case 4), respectively. The BO method and corresponding CRLB in Chapter 8 are introduced to make a comparison, and the obtained position and velocity estimation error curves are shown in Figure 9.12.

For several groups of simulation conditions above, the corresponding estimation errors of the converged position, velocity, and target signal carrier frequency $f_{0}$, as well as the corresponding CRLB, are shown respectively in Table 9.4.

Table 9.3 Position and velocity estimation error of different angle measurement errors in the case of the two-body model when $f_{0}$ is known

| Simulation scenarios/parameters |  | $\sigma_{\beta}=10^{\prime \prime}$ |  |  | $\sigma_{\beta}=150^{\prime \prime}$ |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | Unknown $f_{0}$ | Known $f_{0}$ | CRLB | Unknown $f_{0}$ Known $f_{0}$ CRLB |  |  |  |
| Position estimation error (km) | 0.2105 | 0.1753 | 0.1461 | 3.2094 | 2.3251 | 1.1379 |  |
| Velocity estimation error $(\mathrm{km} / \mathrm{s})$ | $2.24 \times 10^{-4}$ | $2.45 \times 10^{-4}$ | $1.86 \times 10^{-4}$ | 0.0049 | 0.0034 | 0.0014 |  |



Figure 9.11 Position and velocity error curves in the case of the two-body model when $f_{0}$ is known and the angle measurement error is $\sigma_{\beta}=150^{\prime \prime}$. (a) Position estimation error curve, (b) velocity estimation error curve


Figure 9.11 (continued)

According to Figure 9.12 and Table 9.4 for the case of the two-body model in a different frequency error, the accuracy of the passive tracking based on the CM is better than that of the BO method. Though with the increase of frequency measurement error the relative filtering convergence time and the position and velocity estimation errors are increased, their estimation accuracy is still better than that of the BO tracking method, indicating that at the same time with the angle measurement after the frequency measurement information has been increased, even though the frequency measurement accuracy is not high, the estimated performance can also be effectively improved. The basic reason lies in the fact that the observability of the system has been improved after adding frequency measurement information.
With other simulation parameters remaining fixed, the influence of different frequency measurement errors on the performance of passive tracking when adopting the CM and $f_{0}$ is known is tested in this simulation. In the meanwhile, estimation results of the corresponding unknown $f_{0}$ and the CRLB are introduced as a reference (the CRLB at the moment still refers to the estimation error lower bound when $f_{0}$ is known). Calculate the passive tracking estimation error under two conditions of frequency measurement error, that is, under $\sigma_{f}=50 \mathrm{~Hz}$ (Case 5) and $\sigma_{f}=50 \mathrm{kHz}$ (Case 6), the obtained position and velocity estimation error curves are shown in Figure 9.13.


Figure 9.12 Position and velocity error curves of different frequency measurement errors in the case of the two-body model. (a) Position estimation error curve and (b) velocity estimation error curve

For the above two groups of simulation conditions, the corresponding converged position and velocity estimation errors, as well as the corresponding CRLB, are shown respectively in Table 9.5.
According to Figure 9.13 and Table 9.5 for the two-body model, in different frequency errors, the accuracy of passive tracking with the $f_{0}$ known has been improved in comparison with that of the unknown $f_{0}$, but as the frequency increases, the filtering convergence time of the known $f_{0}$ has also greatly increased. According to Table 9.3, relative to the angle measurement error, under different frequency errors the performance improvement of the known $f_{0}$ is poorer than the unknown $f_{0}$.

Table 9.4 Parameter estimation error of different frequency measurement errors in the case of the two-body model

| Simulation scenario/parameter | BO error | $\begin{gathered} \text { Case 1: } \\ \sigma_{f}=50 \mathrm{~Hz} \end{gathered}$ |  | $\begin{gathered} \text { Case 2: } \\ \sigma_{f}=500 \mathrm{~Hz} \end{gathered}$ |  | $\begin{gathered} \text { Case 3: } \\ \sigma_{f}=5 \mathrm{kHz} \\ \hline \end{gathered}$ |  | $\begin{gathered} \text { Case 4: } \\ \sigma_{f}=50 \mathrm{kHz} \end{gathered}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | CM | CRLB | CM | CRLB | CM | CRLB | CM | CRLB |
| Position estimation error (km) | 1.8138 | 0.9916 | 0.6464 | 1.0891 | 1.0446 | 1.4445 | 1.2048 | 1.3996 | 1.3221 |
| Velocity estimation error (km/s) | 0.0020 | 0.0015 | $8.04 \times 10^{-4}$ | 0.0010 | 0.0010 | 0.0015 | 0.0017 | 0.0018 | 0.0018 |
| Target carrier frequency error (Hz) | - | 9.1 | 8.4 | 34.5 | 31.5 | 67.1 | 38.7 | 139.7 | 129.2 |



Figure 9.13 Position and velocity error curves of different frequency errors in the case of the two-body model when $f_{0}$ is known. (a) Position estimation error curve and (b) velocity estimation error curve

Table 9.5 Position and velocity estimation error of different frequency measurement errors in the case of the two-body model when $f_{0}$ is known

| Simulation scenario/parameter | $\sigma_{f}=50 \mathrm{~Hz}$ |  |  | $\sigma_{f}=50 \mathrm{kHz}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Unknown $f_{0}$ | Known $f_{0}$ | CRLB | Unknown $f_{0}$ | Known $f_{0}$ | CRLB |
| Position estimation error (km) | 0.9916 | 0.9086 | 0.4378 | 1.3996 | 1.3421 | 1.3210 |
| Velocity estimation error (km/s) | 0.0015 | 0.0013 | $5.38 \times 10^{-4}$ | 0.0018 | 0.0016 | 0.0018 |



Figure 9.14 Position and velocity error curves in the case of the two-body model when the filtering period is $T=5$ seconds. (a) Position estimation error curve and (b) velocity estimation error curve

### 9.4.1.4 Influences of Those Different Filtering Periods on Passive Tracking Performances

Assume that the initial state errors are the angle measurement error $\sigma_{\beta}=20^{\prime \prime}$, the frequency error $\sigma_{f}=100 \mathrm{~Hz}$, and the target signal carrier frequency $f_{0}=15 \mathrm{GHz}$. Calculate the orbit determination and tracking estimation error when the filtering periods are $T=5$ seconds and $T=50$ seconds respectively. Since the CRLBs are different under different filtering periods, the estimation error curves of different filtering periods are shown as follows:

1. When the filtering period is $T=5$ seconds, the estimation results are shown in Figure 9.14.
2. When the filtering period is $T=50$ seconds, the estimation results are shown in Figure 9.15.

In the above two groups of simulation conditions, the corresponding converged position, velocity, and target signal carrier frequency $f_{0}$ estimation error as well as the corresponding CRLB are shown respectively in Table 9.6.

According to Figure 9.14, Figure 9.15, and Table 9.6, in the case of the two-body model and in different filtering periods, the accuracy of passive tracking based on the CM is clearly better than that of the BO method, and as the filtering period increases, its advantage becomes obvious, and both the filtering convergence time and the estimated accuracy of the position and velocity are much improved. However, as the filtering period increases, the difference between the estimation accuracy of the CM and its relative CRLB curve also increases, indicating that the CM is also sensitive to the filtering period. In the case of the two-body model, since there is a linearization error when adopting the EKF method, after the filtering period increased, the linearization error increases as well and eventually it will result in a decrease in tracking filtering accuracy.

With other simulation conditions remaining fixed, now test the influence of different filtering periods on the performance of passive tracking when adopting the CM and knowing $f_{0}$. In the meanwhile, estimation results of the relative unknown $f_{0}$ and the CRLB are introduced as a reference (the CRLB at the moment still refers to the estimation error lower bound when $f_{0}$ is known). Calculate the passive tracking estimation error under the two conditions of filtering periods $T=5$ seconds and $T=50$ seconds, respectively.
3. When $f_{0}$ is known and the filtering period is $T=5$ seconds, the estimation results are shown in Figure 9.16.
4. When $f_{0}$ is known and the filtering period is $T=50$ seconds, the estimation results are shown in Figure 9.17.

For the above two groups of simulation conditions, the corresponding converged position and velocity estimation errors, as well as the corresponding CRLB, are shown respectively in Table 9.7.

According to Figure 9.16, Figure 9.17, and Table 9.7, in the case of the two-body model and in different filtering periods, the accuracy of passive tracking based on the CM with known $f_{0}$ is clearly improved compared with that of unknown $f_{0}$, and as the filtering period increases, its advantage becomes more and more obvious, and both the filtering convergence time, and accuracy of the position and velocity are clearly improved.


Figure 9.15 Position and velocity error curves in the case of the two-body model when the filtering period is $T=50$ seconds. (a) Position estimation error curve and (b) velocity estimation error curve

Table 9.6 Parameter estimation error of different filtering periods in the case of the two-body model

| Simulation scenario/parameter | $T=5$ seconds |  |  |  |  | $T=50$ seconds |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | BO error | CM error | CRLB |  | BO error | CM error | CRLB |  |
| Position estimation error (km) | 1.1023 | 0.6329 | 0.5211 | 16.3420 | 3.3501 | 1.6484 |  |  |
| Velocity estimation error (km/s) | 0.0013 | $6.26 \times 10^{-4}$ | $6.05 \times 10^{-4}$ | 0.0133 | 0.0080 | 0.0019 |  |  |
| Target carrier frequency error $(\mathrm{Hz})$ | - | 15.9 | 12.3 | - | 249.7 | 41.7 |  |  |



Figure 9.16 Position and velocity error curves in the case of the two-body model when the filtering period is $T=5$ seconds and $f_{0}$ is known. (a) Position estimation error curve and (b) velocity estimation error curve

### 9.4.1.5 Influences of the Ephemeris Error of the Observing Satellite on Passive Tracking Performances

Simulations 1 to 4 in this chapter are all based on a presumption that the ephemeris of the observing satellite has no error. However, in fact the ephemeris of the observer itself cannot be absolutely accurate, so Simulation 5 conducts a stimulation analysis of this issue. Assume that the initial state error is $\left[\begin{array}{lll}300 & 300 & 300 \\ 0.5 & 0.5 & 0.5 \\ 1 e 5\end{array}\right]^{\mathrm{T}}$, the angle measurement error is $\sigma_{\beta}=20^{\prime \prime}$, the frequency measurement error is $\sigma_{f}=100 \mathrm{~Hz}$, the filtering period is $T=10$ seconds, and the target signal carrier frequency is $f_{0}=15 \mathrm{GHz}$. Calculate the passive tracking estimation error under ephemeris error free (Case 1), the position error 200 m and


Figure 9.17 Position and velocity error curves in the case of the two-body model when the filtering period is $T=50$ seconds and $f_{0}$ is known. (a) Position estimation error curve and (b) velocity estimation error curve

Table 9.7 Position and velocity estimation error of different filtering periods in the case of the two-body model when $f_{0}$ is known

| Simulation scenario/parameter | $T=5$ seconds |  |  | $T=50$ seconds |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Unknown $f_{0}$ | Known $f_{0}$ | CRLB | Unknown $f_{0}$ | Known $f_{0}$ | CRLB |
| Position estimation error (km) | 0.6329 | 0.4075 | 0.4035 | 3.3501 | 2.0355 | 1.2378 |
| Velocity estimation error (km/s) | $6.26 \times 10^{-4}$ | $5.28 \times 10^{-4}$ | $5.13 \times 10^{-4}$ | 0.0080 | 0.0049 | 0.0016 |



Figure 9.18 Position and velocity error curves of different ephemeris errors of observing satellites in the case of the two-body model. (a) Position estimation error curve and (b) velocity estimation error curve
velocity error $10 \mathrm{~m} / \mathrm{s}$ (Case 2), and the position error 500 m and velocity error $50 \mathrm{~m} / \mathrm{s}$ (Case 3). The obtained position and velocity estimation error curves are shown in Figure 9.18.
For these groups of simulation conditions, the corresponding estimation errors of the converged position, velocity, and target signal carrier frequency $f_{0}$ are shown in Table 9.8.
According to Figure 9.18 and Table 9.8, in the case of the two-body model, when taking the ephemeris error of the observing satellite itself into consideration, the position and velocity estimation errors do not change much. The variation of the position error is around 10 m and the variation of velocity error is less than $0.1 \mathrm{~m} / \mathrm{s}$, indicating that the ephemeris error of the observing satellite itself has a negligible influence on satellite-to-satellite passive tracking.

Table 9.8 Position and velocity estimation error of different ephemeris errors of observing satellites in the case of the two-body model

| Simulation scenario/parameter | Case 1 | Case 2 | Case 3 |
| :--- | :--- | :--- | :--- |
| Position estimation error $(\mathrm{km})$ | 0.7743 | 0.8125 | 0.8026 |
| Velocity estimation error $(\mathrm{km} / \mathrm{s})$ | $8.0585 \times 10^{-4}$ | $8.7191 \times 10^{-4}$ | $8.6862 \times 10^{-4}$ |
| Target carrier frequency error $(\mathrm{Hz})$ | 21.2 | 21.4 | 21.9 |

### 9.4.2 Simulation Considering $J_{2}$ Perturbation of Earth Oblateness

Previously, in the case of the two-body model and based on the CM method, passive tracking performance of the unknown $f_{0}$ has been analyzed by computer simulation. In the meanwhile,for the three major factors that will have an effect on the observability of the system - the filtering period, the angle measurement error, and the frequency measurement error - the problem of how to improve estimation performance with the known $f_{0}$ case and the unknown $f_{0}$ case adopting the CM is analyzed. Now we continue to study the passive tracking performance of the CM when considering $J_{2}$ perturbation of the earth oblateness.

### 9.4.2 1 Influences of Different Initial State Errors on Passive Tracking Performances

Take the simulated results of the BO method from Chapter 8 as a comparison and suppose that its initial state error is $\left[\begin{array}{lllll}300 & 300 & 300 & 0.5 & 0.5\end{array} 0.5\right]^{\mathrm{T}}$ (Case 1) and that other simulation conditions are identical with those of Simulation 1 in this chapter. Since the CRLB under different initial state errors are identical, here the CRLB is also introduced as the reference of the CRLB (as orbit perturbation has a very small influence on the CRLB, the CRLB of this study in the case of perturbation all adopt the CRLB in the case of the two-body model). The obtained position and velocity estimation error curve is shown in Figure 9.19.
In several groups of the simulation conditions above, the corresponding converged position, velocity, and target signal carrier frequency $f_{0}$ estimation error as well as the corresponding CRLB are shown in Table 9.9.
According to Figure 9.19 and Table 9.9, even though the estimation error adopting the CM has greatly increased in the case of the perturbation condition more than in the case of the two-body model, it still has a performance of position estimation accuracy exceeding 1 km and velocity estimation accuracy exceeding $5 \mathrm{~m} / \mathrm{s}$. In addition, with the increase of the initial state error, the position and velocity estimation errors obtained by the CM have been increased to some extent, and the difference with the CRLB also increases, indicating that in the case of perturbation, the effect of the EKF linearization error by a constant state noise covariance matrix can still be improved.

### 9.4.2.2 Influences of Different Angle Measurement Errors on Passive Tracking Performances

Other simulation conditions are identical with those in Simulation 2 of this chapter. Calculate the CM based on the passive tracking error in the case of perturbation and when the angle


Figure 9.19 Position and velocity error curve of different initial state errors in the case of perturbation. (a) Position estimation error curve and (b) velocity estimation error curve

Table 9.9 Parameter estimation error of different initial state errors in the case of perturbation

| Simulation scenario/parameter | Case 1 | Case 2 | Case 3 | Case 4 | CRLB |
| :--- | :---: | :---: | :---: | :---: | :--- |
| Position estimation error $(\mathrm{km})$ | 4.0558 | 2.7272 | 2.8581 | 2.9378 | 0.3297 |
| Velocity estimation error (km/s) | 0.0179 | 0.0125 | 0.0125 | 0.0126 | $3.5255 \times 10^{-4}$ |
| Target carrier frequency error $(\mathrm{Hz})$ | - | 102.8 | 108.1 | 114.9 | 7.2 |



Figure 9.20 Position and velocity error curve in the case of perturbation when the angle measurement error is $\sigma_{\beta}=50^{\prime \prime}$. (a) Position estimation error curve and (b) velocity estimation error curve
measurement errors are $50^{\prime \prime}$ and $150^{\prime \prime}$, respectively. Here the CRLB and the BO method are also introduced for comparison:

1. When the angle measurement error is $\sigma_{\beta}=50^{\prime \prime}$, the estimation results are shown in Figure 9.20.
2. When the angle measurement error is $\sigma_{\beta}=150^{\prime \prime}$, the estimation results are shown in Figure 9.21.

In the two groups of simulation conditions above, the corresponding converged position, velocity, and the target signal carrier frequency $f_{0}$ estimation error as well as the corresponding CRLB are shown in Table 9.10.

According to Figure 9.21 and Table 9.10, as the angle measurement error increases, the filtering convergence time and the estimation error of position, velocity, and frequency of the CM all increase accordingly, indicating that in the case considering $J_{2}$ perturbation, the estimation accuracy of the CM is clearly better than the BO method. Under the same conditions where the angle measurement error increases from $50^{\prime \prime}$ to $150^{\prime \prime}$, the position estimation error of the BO method increases about 1.8 km , and the velocity error increases about $2 \mathrm{~m} / \mathrm{s}$, but the position estimation error of the CM only increases about 0.3 km and the velocity error increases $1 \mathrm{~m} / \mathrm{s}$, this indicates that if the angle measurement accuracy is going to worsen, the same size and performance deterioration of the CM is much less than the BO method.


Figure 9.21 Position and velocity error curve in the case of perturbation when the angle measurement error is $\sigma_{\beta}=150^{\prime \prime}$. (a) Position estimation error curve and (b) velocity estimation error curve

Table 9.10 Parameter estimation error of different angle measurement errors in the case of $J_{2}$ perturbation

| Simulation scenario/parameter | $\sigma_{\beta}=50^{\prime \prime}$ |  |  |  |  | $\sigma_{\beta}=150^{\prime \prime}$ |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | BO error CM error | CRLB |  | BO error | CM error | CRLB |  |  |
| Position estimation error $(\mathrm{km})$ | 5.0624 | 3.6037 | 0.7026 |  | 6.8377 | 3.9386 | 1.8198 |  |
| Velocity estimation error $(\mathrm{km} / \mathrm{s})$ | 0.0185 | 0.0146 | $8.83 \times 10^{-4}$ | 0.0209 | 0.0154 | 0.0022 |  |  |
| Target carrier frequency error $(\mathrm{Hz})$ | - | 97.3 | 11.4 | - | 206.3 | 21.9 |  |  |



Figure 9.22 Position and velocity error curve in the case of perturbation when $f_{0}$ is known and the angle measurement error $\sigma_{\beta}=50^{\prime \prime}$. (a) Position estimation error curve and (b) velocity estimation error curve

With other simulation conditions fixed, the increased filtering period is $T=5$ seconds. Test the influence that different angle measurement errors have on passive tracking performances by the CM when $f_{0}$ is known. Here the relative unknown $f_{0}$ estimation results and the CRLB are still introduced as the reference.
3. When $f_{0}$ is known and the angle measurement error is $\sigma_{\beta}=50^{\prime \prime}$, the estimation result is shown in Figure 9.22.
4. When $f_{0}$ is known and the angle measurement error is $\sigma_{\beta}=150^{\prime \prime}$, the estimation result is shown in Figure 9.23.

In the two groups of simulation conditions above, the corresponding converged position and velocity estimation errors as well as the corresponding CRLB are shown in Table 9.11.

According to Figure 9.22, Figure 9.23 and Table 9.11, under the condition of different angle measurement errors when considering $\mathrm{J}_{2}$ orbit perturbation, the accuracy of passive tracking based on the CM with the $f_{0}$ known improved to some extent in comparison with the $f_{0}$ unknown case, but this shows that the position estimation accuracy, the filtering convergence time, and the velocity estimation accuracy has barely improved with the increase of angle measurement error. Its performance improvement has become smaller relative to the unknown $f_{0}$ conditions.

### 9.4.2.3 Influences of Different Frequency Measurement Errors on Passive Tracking Performances

The simulation conditions are identical with those in Simulation 3 of this chapter. Calculate the passive tracking error under different frequency measurement errors in the case of perturbation. Here the BO method and the relative CRLB are also introduced as a comparison and the position and velocity estimation error curves are shown in Figure 9.24.
In several groups of simulation conditions above, the corresponding converged position, velocity, and the target signal carrier frequency $f_{0}$ estimation error as well as the corresponding CRLB are shown in Table 9.12.
According to Figure 9.24 and Table 9.12, in the case of $J_{2}$ perturbation, the accuracy of the passive tracking based on the CM is better than that of the BO method under different frequency measurement errors. As the frequency measurement error increases, the filtering convergence time, position error, and velocity estimation error all increase. However, even though the frequency measurement error increases to $\sigma_{f}=50 \mathrm{kHz}$, its position estimation accuracy still has a nearly 100 m improvement compared with the BO tracking method, which indicates that

Table 9.11 Position and velocity estimation error of different angle measurement errors in the case of the two-body model when $f_{0}$ is known

| Simulation scenario/parameter | $\sigma_{\beta}=50^{\prime \prime}$ |  |  | $\sigma_{\beta}=150{ }^{\prime \prime}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Unknown $f_{0}$ | Known $f_{0}$ | CRLB | Unknown $f_{0}$ | Known $f_{0}$ | CRLB |
| Position estimation error (km) | 4.0929 | 2.3666 | 0.6352 | 6.6180 | 4.3861 | 1.3919 |
| Velocity estimation error (km/s) | 0.0118 | 0.0122 | $6.32 \times 10^{-4}$ | 0.0088 | 0.0116 | 0.0014 |



Figure 9.23 Position and velocity error curve in the case of perturbation when $f_{0}$ is known and the angle measurement error is $\sigma_{\beta}=150^{\prime \prime}$. (a) Position estimation error curve and (b) velocity estimation error curve
after the frequency measurement accuracy has been improved, the estimation accuracy will be better than that of the BO method, even though the frequency accuracy is not high.

### 9.4.2.4 Influences of Those Different Filtering Periods on Passive Tracking Performances

The simulation conditions are the same as those of Simulation 4 in this chapter. Calculate the error of the satellite-to-satellite passive tracking estimation when the filtering periods are $T=20$ seconds and $T=50$ seconds, respectively. Since the CRLBs are different under


Figure 9.24 Position and velocity error curve of different frequency measurements in the case of perturbation. (a) Position estimation error curve and (b) velocity estimation error curve

Table 9.12 Estimation error in the case of $J_{2}$ perturbation

| Simulation scenario/parameter | BO error | Case 1:$\sigma_{f}=50 \mathrm{~Hz}$ |  | $\begin{gathered} \text { Case 2: } \\ \sigma_{f}=500 \mathrm{~Hz} \end{gathered}$ |  | $\begin{gathered} \text { Case 3: } \\ \sigma_{f}=5 \mathrm{kHz} \end{gathered}$ |  | $\begin{gathered} \text { Case 4: } \\ \sigma_{f}=50 \mathrm{kHz} \end{gathered}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | CM | CRLB | CM | CRLB | CM | CRLB | CM | CRLB |
| Position estimation error (km) | 5.0624 | 3.593 | 330.6464 | 3.6218 | 1.0446 | 3.6353 | 1.2048 | 4.9376 | 1.3221 |
| Velocity estimation error (km/s) | 0.0185 | 0.01 | $458.04 \times 10^{-4}$ | 0.0146 | 0.0010 | 0.0149 | 0.0017 | 0.0180 | 0.0018 |
| Target carrier frequency error (Hz) | - | 97.4 | 8.4 | 100.4 | 31.5 | 150.1 | 38.7 | 224.9 | 129.2 |



Figure 9.25 Position and velocity error curve in the case of perturbation when the filtering period is $T=20$ seconds. (a) Position estimation error curve and (b) velocity estimation error curve
different filtering periods, the estimation error curves in different filtering periods are shown in Figures 9.25 and 9.26:

1. When the filtering period is $T=20$ seconds, the estimation results are as shown in Figure 9.25.
2. When the filtering period is $T=50$ seconds, the estimation results are as shown in Figure 9.26.

In the two groups of simulation conditions above, the corresponding converged position, velocity, and the target signal carrier frequency $f_{0}$ estimation error as well as the corresponding CRLB are shown in Table 9.13.


Figure 9.26 Position and velocity error curve in the case of perturbation when the filtering period is $T=50$ seconds. (a) Position estimation error curve and (b) velocity estimation error curve

Table 9.13 Parameter estimation error of different filtering periods in the case of perturbation

| Simulation scenario/parameter | $T=20$ seconds |  |  |  |  | $T=50$ seconds |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | BO error | CM error | CRLB |  | BO error | CM error | CRLB |  |
| Position estimation error (km) | 6.1300 | 3.7600 | 1.0415 |  | 11.9188 | 3.3206 | 1.6484 |  |
| Velocity estimation error $(\mathrm{km} / \mathrm{s})$ | 0.0203 | 0.0140 | 0.0012 |  | 0.0132 | 0.0098 | 0.0019 |  |
| Target carrier frequency error (Hz) | - | 128.9 | 25.2 |  | - | 90.1 | 41.7 |  |

According to Figure 9.25, Figure 9.26, and Table 9.13, in the case of perturbation and with different filtering periods, the accuracy of the passive tracking based on the CM is clearly better than that of the BO method. As the filtering period increases, its advantage becomes more and more obvious, and both the filtering convergence time and the estimated accuracy of the position and velocity are much improved. However, when the filtering period $T=50$ seconds, the location accuracy obtained from the CM is much higher than that of $T=20$ seconds, which does not coincide with the conclusion obtained from the observability analysis in previous systems, indicating that the linearization error in the case of $\mathrm{J}_{2}$ perturbation is no longer stable and the EKF by the constant state noise covariance matrix cannot meet the requirement of high accuracy.


Figure 9.27 Position and velocity error curve in the case of perturbation when the filtering period is $T=20$ seconds and $f_{0}$ is known. (a) Position estimation error curve and (b) velocity estimation error curve


Figure 9.28 Position and velocity error curve in the case of perturbation when the filtering period is $T=50$ seconds and $f_{0}$ is known. (a) Position estimation error curve and (b) velocity estimation error curve

Table 9.14 Position and velocity estimation error of different filtering periods in the case of perturbation when $f_{0}$ is known

| Simulation <br> scenario/parameter | $T=20$ seconds |  |  |  |  | $T=50$ seconds |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Unknown $f_{0}$ | Known $f_{0}$ | CRLB |  | Unknown $f_{0}$ | Known $f_{0}$ | CRLB |  |
| Position estimation <br> error (km) | 4.9157 | 2.0961 | 0.5775 |  | 5.3692 | 3.5952 | 0.9173 |  |
| Velocity estimation <br> error (km/s) | 0.0100 | 0.0117 | $5.82 \times 10^{-4}$ | 0.0076 | 0.0078 | $9.1191 \times 10^{-4}$ |  |  |

With other simulation conditions fixed, if the CM is used in the $f_{0}$ known case, then the influence of different filtering periods on the performance of passive tracking can be analyzed. In order to compare the performances, the estimation results of the unknown $f_{0}$ and the CRLB are still introduced as references (the CRLB refers to the CRLB when $f_{0}$ is known).
3. When $f_{0}$ is known, the filtering period is $T=20$ seconds and the estimation results are shown in Figure 9.27.
4. When $f_{0}$ is known, the filtering period is $T=50$ seconds and the estimation results are shown in Figure 9.28.

In the two groups of simulation conditions above, the corresponding converged position and velocity estimation errors as well as the corresponding CRLB are shown in Table 9.14.

According to Figure 9.27, Figure 9.28, and Table 9.14, when considering different filtering periods and adopting the $\mathbf{C M}$, the passive tracking accuracy when $f_{0}$ is known is better compared with the $f_{0}$ unknown case, but is similar to that of the influence of angle measurement error analysis. This improvement only reflects on the position estimation accuracy and filtering convergence time, but the velocity estimation accuracy has barely improved.

### 9.5 Summary

Through the results of theoretical analysis and simulation, the following conclusions can be made:

1. The influences of the factors on the performance of passive tracking based on the CM in descending order are: filtering period, angle measurement error, frequency measurement error, initial state error, and ephemeris error of the observer itself.
2. Considering only the two-body model, the estimation performance of the CM is close to optimal, that is, it is very close to the CRLB, and this method is clearly superior to the BO method in Chapter 8. This indicates that when two satellites run in different orbits, adding the target frequency information on the basis of angle measurement can effectively improve the observability of the system and location accuracy.
3. Taking perturbation of the satellite into consideration, the performance of the CM clearly becomes worse in comparison with its performance in the two-body model, and it obviously cannot reach the CRLB, indicating that the linearization errors caused by the EKF method are more complex than those in the two-body model and the constant state noise covariance matrix is not enough to correct the linearization error from the nonlinear state equation. In addition, under the perturbation condition, the tracking performance of the CM in the two-body model is clearly superior to that of the BO tracking method.
4. In different filtering periods and angle measurement errors of the two-body model, the estimation performance in the $f_{0}$ known case will be improved compared with the $f_{0}$ unknown case. When taking different filtering periods and angle measurement errors into consideration, the performance of location estimation accuracy and filtering convergence time in the $f_{0}$ known case has also been improved compared with the $f_{0}$ unknown case, but the corresponding velocity estimation accuracy has barely improved. In addition, under different frequency measurement errors, the location performance of the known $f_{0}$ case is better than that of the unknown $f_{0}$ case, but relative to filtering periods and angle measurement
errors, its performance improvement is much less. In general, when adopting the CM, if part of the prior information for the target can be introduced, the tracking performance of the algorithm can be further improved.

## References

1. Chan, Y. T. and Rudnicki, S. (1992) Bearings-only and Doppler-bearing tracking using instrumental variables. IEEE Transactions on Aerospace and Electronic Systems, 28(4), 1076-1082.
2. Becker, K. (1992) An efficient method of passive emitter location. IEEE Transactions on Aerospace and Electronic Systems, 28(4), 1091-1104.
3. Becker, K. (2005) Three-dimensional target motion analysis using angles and frequency measurements. IEEE Transactions on Aerospace and Electronic Systems, 41(1), 284-301.
4. Becker, K. (1996) A general approach to TMA observability from angle and frequency measurements. IEEE Transactions on Aerospace and Electronic Systems, 32(1), 487-494.
5. Wang, Y. and Liu, Y. (2003) Military Satellite and Application Concept. Beijing: National Defence Industry Press (in Chinese).
6. Xi, X., Wang, W., and Gao, Y. (2003) Fundamentals of Near-Earth Spacecraft Orbit. Changsha: NUDT Publish House (in Chinese).
7. Bar-Shalom, Y., Li, R.X. and Kjrubarajan, T. (2001) Estimation with Applications to Tracking and Navigation. New York: John Wiley \& Sons, Inc.
8. Zhang, X. (2004) Matrix Analysis and Applications. Beijing: Tshinghua University Press, p. 9.
9. Liu, Z. and Chen, Z. (2004) Application of condition number in observability analysis of system. Journal of System Simulation, 16(7): P1552-P1555 (in Chinese).
10. Guo, F. and Fan, Y. (2005) A tracking method for satellite-to-satellite passive localization in space information confrontation. Journal of Astronautics, 26(2): P196-P200 (in Chinese).

## 10

## Satellite-to-Satellite Passive Orbit Determination Based on Frequency Only

Chapters 8 and 9 focused on the methods for satellite-to-satellite passive tracking with bearings only (BO) and with bearings and frequency information, respectively. They both used the position and velocity vector of the target satellite in the J2000.0 coordinates system as the state variable to be estimated and the extended Kalman filter (EKF) algorithm was used to achieve passive tracking of the target satellite. In general, the researches in the previous two chapters both used the satellite as a moving target for system modeling and employed the nonlinear filtering algorithm to achieve passive tracking of the satellite.
A satellite is a maneuvering target that moves under a special law of motion (i.e., Kepler's law). In the case of the two-body model, the satellite orbit can be a closed elliptical or circular orbit, and its motion state can be well described by the orbit elements at the epoch time. Thus the passive tracking of the satellite target can be converted to parameter estimation of the orbit elements at the epoch time.
By selecting the observed parameters and treating the orbit elements of the target satellite at the epoch time as the parameter to be estimated, a constraint relation can be established between the measurement information and the orbit elements of the target satellite at the epoch time, so as to acquire the cost function to be optimized (minimized or maximized). Eventually, some nonlinear estimation algorithms should be employed to achieve parameter estimation of the orbit elements of the target satellite at the epoch time. Here it refers to the passive orbit determination of the target satellite. For the issue to be researched in this chapter, the epoch time is the time of starting the observation. If the observation starts at the time $k_{0}$ and ends at the time $k_{0}+N-1$, the passive orbit determination is to determine the orbit elements of the target satellite at the time $k_{0}$ using the measurement information from $k_{0}$ to $k_{0}+N-1$.

There are two key problems in the nonlinear state localization system: the first is the selection of the observed parameter and the second is the nonlinear estimation algorithm.

1. For the selection of the observed parameter, reference [1] researched into a method for satellite-to-satellite frequency that only based passive localization of the ground (two-dimensional) or the aerial (three-dimensional) target by frequency-only measurement. With the help of the target frequency information received by the satellite at $N$ different moments, a nonlinear cost function that contains the unknown emitter signal frequency and the target position is eventually acquired. By solving this nonlinear function the estimated position of the emitter can be obtained. Thus, in this chapter, the Doppler frequency information of reference [1] has been selected as the observed parameter and estimation of the orbit elements of the target satellite at the epoch time is modeled into a single-target nonlinear optimization issue.
2. The nonlinear estimation algorithms for solving optimization problems include the following types: the first type is to acquire the analytic solution in closed form [2]; the second type is the grid search method [3]; and the third type is the iterative search method [4]. Iterative computation is involved (e.g., solution of the eccentric argument of perigee $E$ ) when the orbit elements at the epoch time are converted to the position and velocity of the satellite. Thus there is no analytic solution for using the problem of Doppler frequency to determine the position and velocity or the orbit elements of the satellite at the epoch time. In addition, the orbit elements at the epoch time, which is at least in a six-dimensional state, using the grid search method may involve a large amount of computational load and is therefore inadvisable for the research in this chapter. The general iterative search method has an inherent disadvantage that it is greatly dependent on the initial value and most of the algorithms, such as the Newton iteration method and the gradient method, require calculation of the gradient of the cost function [5]. In recent years, Kennedy and Eberhart proposed a particle swarm optimization (PSO) algorithm in 1995 [6-9], which is a swarm intelligent optimization algorithm. It originated from the research into the group behavior of bird flock and fish shoal. This algorithm has intense parallelism and does not require gradient information; instead, it uses only the value information of the target, and therefor is quite commonly used [9]. The PSO algorithm has been proven to be effective for numerous actual applications and has drawn extensive attention from numerous domestic and foreign scholars [6-8]. Therefore, this chapter introduces the PSO algorithm for orbit determination of satellite and researches into the method for satellite-to-satellite passive orbit determination.

This chapter will also research into the observability of the satellite-to-satellite passive orbit determination system based on frequency-only measurement. The orbit elements of the target satellite at the epoch time are used as the parameter to be estimated. Thus the relation between the system observability and the orbit elements of the target satellite under certain observation geometric conditions is researched in this chapter.

### 10.1 The Theory and Mathematical Model of Passive Orbit Determination Based on Frequency Only

### 10.1.1 The Theory of Orbit Determination Based on Frequency Only

Similar to the one in Chapter 9, the satellite target to be estimated in two cases is researched in this chapter: the unknown signal carrier frequency $f_{0}$ case and the known signal carrier frequency $f_{0}$ case.
For a noncooperative target satellite, there is relative movement between two satellites because they are not on the same orbit. If the Doppler frequency information between satellites can be measured, there is a nonlinear relation between the Doppler frequency and the position and velocity components of two satellites and the signal carrier frequency $f_{0}$ of the target satellite. If the orbit elements of the target satellite at the epoch time are the parameter to be estimated, either in the case of the two-body model or with consideration of orbit perturbation, orbit extrapolation can be achieved for this satellite, in order to acquire the position and velocity vector of the satellite at any time $t$. With the assumption that the ephemeris of the observing satellite are known, a more complicated nonlinear relation can be established between the measured Doppler frequency information and the orbit elements of the target satellite at the epoch time. Establish a single-target function and let it be optimized (minimized or maximized) based on the Doppler frequency. By iterative optimization of the established cost function, the orbit elements of the target satellite at the epoch time can be estimated, that is, the passive orbit determination of the target satellite can be achieved by measuring the Doppler frequency only. What is described above is the theory of satellite-to-satellite passive orbit determination by Doppler frequency only.

### 10.1.2 The System Model in the Case of the Two-Body Model

In this chapter, the state parameters to be estimated are the orbit elements of the target satellite at the epoch time, which is similar to the state variables selected in reference [1]. They include the semi-major axis $a$, eccentricity $e$, orbit inclination $i$, RAAN (right ascension of the ascending node) $\Omega$, the argument of perigee $\omega$, and the epoch mean argument of perigee $M_{0}$, the latter parameter being the last one to be estimated, unlike the time past perigee $\tau$, which was the last one in reference [1]. These two parameters are the same essentially, which is the description of satellite perigee, and can be mutually converted as in the conversion expression in Section 2.2.2, but they are in different units and their range of values are also different. The time past perigee $\tau$ is in seconds and is not bounded; the epoch mean argument of perigee $M_{0}$ is in radian, for one orbit period corresponds to $2 \pi$, and so it has a fixed range of values $[0,2 \pi)$. For the issue to be researched in this chapter, the iterative search method is needed to estimate the state parameter and the parameter to be estimated preferably has a fixed range of values so that the initial value can be easily selected. This means that the epoch mean argument of perigee $M_{0}$ is the last parameter to be estimated here. Therefore, the state parameter $\boldsymbol{X}_{F O}$ to be estimated in this chapter is (in order to be different from the state variables in previous
chapters, the state parameter to be estimated with a frequency measurement only is marked as $\boldsymbol{X}_{F O}$ ):

$$
\boldsymbol{X}_{F O}=\left[\begin{array}{llllll}
a & e & i & \Omega & \omega & M_{0} \tag{10.1}
\end{array}\right]^{\mathrm{T}}
$$

In this expression, the semi-major axis $a$ and eccentricity $e$ determine the size and shape of the elliptic orbit of the satellite, the orbit inclination $i$ and RAAN $\Omega$ determine the position of the orbital plane in the space, the argument of perigee $\omega$ determines the bearing of the ellipse in the orbital plane, and $M_{0}$ determines the mean argument of perigee of the satellite at time $t=0$ [10].
In the case of the two-body problem, the orbit elements of the target satellite are constant; thus the position and velocity component of the satellite at any time $t$ can both be acquired by extrapolation according to the orbit elements of the satellite at the epoch time. The calculation expression is seen in the part for satellite ephemeris calculation in the case of the two-body model in Section 2.2.1.
With the assumption that the position and velocity component of the observing satellite ( O ) at the $i$ th frequency measurement are $\boldsymbol{r}_{O}^{i}$ and $\dot{\boldsymbol{r}}_{O}^{i}$, respectively, now the position and velocity component of the target satellite (T) are $\boldsymbol{r}_{T}^{i}$ and $\dot{\boldsymbol{r}}_{T}^{i}$. Due to the relative movement between the two satellites, the frequency value of the observing satellite measured at the $i$ th time is expressed as

$$
\begin{equation*}
f_{\mathrm{m}}^{i}=f_{0}\left[1-\frac{\left(\dot{\boldsymbol{r}}_{i} \cdot \boldsymbol{r}_{i}\right)}{c r_{i}}\right]+\varepsilon_{i} \tag{10.2}
\end{equation*}
$$

where $c$ is the propagation speed of the electromagnetic wave in free space, $f_{0}$ is the signal carrier frequency, $\varepsilon_{i}$ is the random error in frequency measurement, assuming all errors are i.i.d. (independent and identically distributed), and the zero-mean Gaussian is distributed with variance of $\sigma_{f}^{2}$. In addition, $\boldsymbol{r}_{i}=\boldsymbol{r}_{T}^{i}-\boldsymbol{r}_{O}^{i}, \dot{\boldsymbol{r}}_{i}=\dot{\boldsymbol{r}}_{T}^{i}-\dot{\boldsymbol{r}}_{O}^{i}$, and $r_{i}$ are respectively the relative position, velocity, and the relative distance between satellites.
When $i=1, \ldots, N$, there are $N$ frequency measurements in total and expression (10.2) can be expressed as

$$
\begin{equation*}
\boldsymbol{Z}_{m}=f_{0} \cdot \boldsymbol{H}_{0}\left(\boldsymbol{r}_{T}, \dot{\boldsymbol{r}}_{T}\right)+\varepsilon=f_{0} \cdot \boldsymbol{H}_{F O}\left(\boldsymbol{X}_{F O}\right)+\boldsymbol{\varepsilon} \tag{10.3}
\end{equation*}
$$

where $\boldsymbol{Z}_{m}=\left[\begin{array}{llll}f_{m}^{1} & f_{m}^{2} & \ldots & f_{m}^{N}\end{array}\right]^{\mathrm{T}}$ is the Doppler frequency measurement vector, $\boldsymbol{\varepsilon}=\left[\begin{array}{lll}\varepsilon_{1} & \varepsilon_{2} & \ldots\end{array}\right.$ $\left.\varepsilon_{N}\right]^{\mathrm{T}}$ is the measurement error vector, $\boldsymbol{r}_{T}=\left[\begin{array}{llll}\boldsymbol{r}_{T}^{1} & \boldsymbol{r}_{T}^{2} & \ldots & \boldsymbol{r}_{T}^{N}\end{array}\right]^{\mathrm{T}}$ and $\dot{\boldsymbol{r}}_{T}=\left[\begin{array}{llll}\dot{\boldsymbol{r}}_{T}^{1} & \dot{\boldsymbol{r}}_{T}^{2} & \ldots & \dot{\boldsymbol{r}}_{T}^{N}\end{array}\right]^{\mathrm{T}}$ are the position and velocity component of the target satellite, respectively, $\boldsymbol{H}_{0}(\cdot)$ and $\boldsymbol{H}_{F O}(\cdot)$ are respectively the nonlinear transformation of the state variable $\boldsymbol{X}_{F O}$ by the expression (10.2), and the subscript ' $m$ ' in Equations (10.2) and (10.3) is the parameter measurement value. There is only one transform of the state parameter to be estimated, $\boldsymbol{X}_{F O}$, to the position and velocity vector each time [10], so in expression (10.3), the position and velocity vector of the target satellite can be substituted by the state parameter to be estimated, $\boldsymbol{X}_{F O}$.
By now, the position $\boldsymbol{r}_{O}$ and velocity vector $\dot{\boldsymbol{r}}_{O}$ of the observing satellite $(\mathrm{O})$ in expression (10.3) are already known; $\boldsymbol{Z}_{\mathrm{m}}$ is the frequency measurement vector; the only state parameter to be estimated, $\boldsymbol{X}_{F O}$, is unknown (for the case of unknown signal carrier frequency $f_{0}, f_{0}$ is also unknown); and by solving the nonlinear expression (10.3), the state parameter to be estimated, $\boldsymbol{X}_{F O}$, can be acquired.

### 10.1.3 The System Model for $J_{2}$ Perturbation of Earth Oblateness

Similar to the analyses in the previous two chapters, since it is too idealistic to analyze a satellite-to-satellite passive orbit determination problem under the two-body model, this section will take into account the effects of orbit perturbation on the passive orbit determination problem.
The system model considering $J_{2}$ perturbation of the earth oblateness is, in form, similar to the state model to be estimated, $\boldsymbol{X}_{F O}$, and measurement Equation (10.3) for the previous analysis under the two-body model system. The only difference is the way used to extrapolate the orbit elements of the satellite at the epoch time to the position and velocity vector at any time $t$.
In the case of the two-body model, the orbit elements of the satellite are fixed and so the epoch time can be directly extrapolated to acquire the position and velocity of the satellite at any time $t[11,12]$. However, with consideration of perturbation, the orbit elements of the satellite no longer form a constant, but keep changing all the time. Direct extrapolation using the orbit elements at the epoch time by the expression for the two-body model will not describe the motion state of the satellite precisely and therefore, as the extrapolation time increases, the error increases rapidly. To solve the problem of orbit estimation under the perturbation condition, two methods for satellite perturbation motion can be used [10, 13].
The first method is the analytical solution. This method, based on the fact that the perturbative force is small compared with the gravity at the center of mass of the earth, expands the perturbation motion expression into a series and then integrates the motion expression within a certain accuracy range, thus establishing the analytic function relation where the orbit elements change with time. In other words, the orbit elements at the epoch time need to be extrapolated to any time $t$ considering $J_{2}$ perturbation, to acquire the instantaneous orbit elements at this time. Then the orbit elements will be converted according to the two-body model to calculate the corresponding position and velocity vector.
The second method is the numerical solution. This method takes the position and velocity vector of the satellite at a given epoch time $t_{0}$ as the initial values and then employs the numerical method to acquire the precise position and velocity vector of the satellite at the corresponding time $t$. In other words, the orbit elements at the epoch time will first be converted according to the two-body model to calculate the corresponding position and velocity vector, use them as the initial values, and then numerical values are integrated according to the perturbation motion expression to acquire the position and velocity vector of the satellite at any time $t$.
The calculation steps of the analytical solution method and numerical solution method for orbit extrapolation with consideration of $J_{2}$ perturbation are given below:

1. The analytical solution for solving the perturbation motion equation [13]

Suppose the orbit elements of the satellite at the epoch time $t_{0}$ are known to be $\sigma\left(t_{0}\right)$, which is the initial condition. Solve the Lagrange planetary motion equation [10] to obtain the orbit elements variation $\Delta \sigma(t)$ from $t_{0}$ to any time $t$, in order to obtain the instantaneous orbit elements $\sigma(t)=\sigma\left(t_{0}\right)+\Delta \sigma(t)$ at time $t$. The analytical solution method usually uses the series solution. The perturbation is relatively small, so the function, on the right of the

Lagrange planetary motion equation, which contains the orbit elements, can be expanded into a series, as the approximate value of $\sigma$, to obtain a solution of certain accuracy.

When the $J_{2}$ perturbation of the earth oblateness is considered, it is called the first-order approximate solution of the series method. Then the perturbation function $R$ can be decomposed as follows [10]:

$$
\begin{equation*}
R=R_{c}^{(1)}+R_{s}^{(1)}+R_{l}^{(1)} \tag{10.4}
\end{equation*}
$$

where $R_{c}^{(1)}$ is the first-order secular term, $R_{s}^{(1)}$ is the first-order short-period term, and $R_{l}^{(1)}$ is the first-order long-period term. Solve the above components to acquire the solution to the satellite perturbation motion with consideration of $J_{2}$ perturbation:

$$
\begin{equation*}
\sigma(t)=\sigma\left(t_{0}\right)+\Theta \cdot M_{0}\left(t-t_{0}\right)+\sigma_{1}\left(t-t_{0}\right)+\sigma_{l}^{(1)}(t)+\sigma_{s}^{(1)}(t), \tag{10.5}
\end{equation*}
$$

where $\Theta=\left[\begin{array}{llllll}0 & 0 & 0 & 0 & 0 & 1\end{array}\right]^{\mathrm{T}}$. The known initial conditions are the orbit elements $\sigma\left(t_{0}\right)=$ $\left[\begin{array}{lllll}a_{0} & e_{0} & i_{0} & \Omega_{0} & \omega_{0}\end{array} M_{0}\right]^{\mathrm{T}}$ of the satellite at epoch $t_{0}$ time. The calculating steps for obtaining the position and velocity vector of the satellite at any time $t$ with consideration of $J_{2}$ perturbation oblateness are given in Table 10.1.

Table 10.1 gives the calculation steps for the series solution with consideration of $J_{2}$ perturbation of the earth oblateness. The detailed series solution for perturbation motion can be found in reference [10].
2. The numerical solution method for solving the perturbation motion equation [14]

With the known satellite orbit elements $\sigma\left(t_{0}\right)$ at the epoch time $t_{0}$, use the satellite ephemeris calculation expression in the case of the two-body model in the Appendix to convert $\sigma\left(t_{0}\right)$ to the position and velocity vector of the satellite at the time $t_{0}$, which are the initial values, and use the fourth-order Runge-Kutta method to integrate the values (if there is a higher requirement for accuracy, the Runge-Kutta of higher order can be used, e.g., RKF5(6) or RKF7(8)) to obtain the position and velocity vector of the satellite at any time target. The Runge-Kutta method and high-order RK method can be found in reference [15] and will not be listed here in detail. According to the tests, in the application of passive orbit determination of the satellite, the fourth-order Runge-Kutta method already provides adequate accuracy for numerical integration.
3. The comparison between the two solution methods for solving perturbation motion equation

The analytic method obtains the solution on the basis of the orbit elements of the satellite. This method has explicit significance, but it is more complicated. It is suitable for applications such as analysis on and research into the law of motion of the satellite or orbit design. The solution of the numerical method is easy to obtain and is easier to understand. However, the solving process does not involve the orbit elements of the satellite and so it is difficult to obtain the regular patterns concerning satellite orbits. This method is more suitable for precise orbit determination of the satellite.

Therefore these two solution methods both have advantages and disadvantages and are both suitable for specific applications. This chapter only intends to extrapolate the position and velocity of the satellite at any time $t$ according to the orbit elements at the epoch time under perturbation. Therefore the numerical solution method will be selected to solve the orbit determination problem under perturbation.

Table 10.1 Calculation steps to find the satellite ephemeris by the analytic method with consideration of $J_{2}$ term perturbation of the earth oblateness

1. Calculate the first-order short-period term $\sigma_{s}^{(1)}\left(t_{0}\right)$ and the first-order long-period term $\sigma_{l}^{(1)}\left(t_{0}\right)$ at the epoch time $t_{0}$
2. Calculate the mean element $\bar{\sigma}\left(t_{0}\right)$ at the epoch time $t_{0}$ :

$$
\bar{\sigma}\left(t_{0}\right)=\sigma\left(t_{0}\right)-\sigma_{s}^{(1)}\left(t_{0}\right)-\sigma_{l}^{(1)}\left(t_{0}\right)
$$

3. Use the mean element $\bar{\sigma}\left(t_{0}\right)$ at the epoch time $t_{0}$ to calculate the first-order secular term $\sigma_{1}\left(t-t_{0}\right)$
4. Calculate the mean element $\bar{\sigma}(t)$ at the time $t$ to be solved:

$$
\bar{\sigma}(t)=\bar{\sigma}\left(t_{0}\right)+\Theta \cdot M_{0}\left(t-t_{0}\right)+\sigma_{1}\left(t-t_{0}\right)
$$

5. Calculate the first-order short-period term $\sigma_{s}^{(1)}(t)$ and the first-order secular term $\sigma_{l}^{(1)}(t)$ at the time $t$ to be solved
6. Use the expression (10.5) to calculate the instantaneous orbit elements $\sigma(t)$ at the time to be solved $t$
7. Use the satellite ephemeris transformation in the case of the two-body model to convert the instantaneous orbit elements $\sigma(t)$ at the time to be solved to the position and velocity vector at the corresponding time

### 10.2 Satellite-to-Satellite Passive Orbit Determination Based on PSO and Frequency

The previous section analyzed the theory and mathematical model of satellite-to-satellite passive orbit determination using frequency only. This section will focus on the methods for solving nonlinear Equation (10.3). According to the analysis above, a highly nonlinear relation exists between the state parameter to be estimated, $\boldsymbol{X}_{F O}$, and the observed parameter. frequency. So the iterative method can be involved as the analytic method may not be feasible for solving Equation (10.3). In addition, the grid search method requires such a large amount of calculation that it is impossible to achieve a six-dimensional search and the fact that the calculation result is not unique makes the grid search method infeasible for solving Equation (10.3). This section will focus on the research into the iterative method for solving the nonlinear Equation (10.3). Here the particle swarm optimization (PSO) algorithm is introduced as a method for state estimation of $\boldsymbol{X}_{F O}$.

### 10.2.1 Introduction of Particle Swarm Optimization (PSO)

The PSO algorithm was inspired by the foraging behavior of bird flocks and is essentially an evolutionary algorithm (EA) based on swarm intelligence. It applies the position-velocity search mode [6]. Every particle has two properties: position and velocity. The position of a particle is a candidate solution in the solution space; the degree of superiority of the solution is determined by the fitness function; and the calculation of the fitness function is defined by the cost function to be optimized. The velocity of a particle determines the moving step of the particle in the solution space at every iteration [7].

PSO is first randomly initialized into a swarm of particles in the solution space. With the assumption that the number of particles is $P$ and the solution space is $L$-dimensional, the position of the particle $i$ in the $L$-dimensional solution space is expressed as $\boldsymbol{x}_{i}=\left(x_{i 1}, x_{i 2}, \ldots, x_{i L}\right)$ and its velocity as $\boldsymbol{v}_{i}=\left(v_{i 1}, v_{i 2}, \ldots, v_{i L}\right)$. In every iteration, the particle upgrades its velocity component and position component by dynamically tracking the two extreme values. Among two extreme values, one is the optimal solution that has been found by the particle from the initialization to the current iteration, that is, $\boldsymbol{p} \boldsymbol{B e s t} \boldsymbol{t}_{i}=\left(p_{i 1}, p_{i 2}, \ldots, p_{i L}\right)$, and the other is the globally optimal solution that has been found by the swarm of particles until the current iteration in history, that is, gBest $=\left(g_{1}, g_{2}, \ldots, g_{L}\right)$. If the superscript ' $n$ ' is the number of iterations, the updating expression for the velocity and position of particle $i$ is expressed as

$$
\left.\begin{array}{rl}
\boldsymbol{v}_{i}^{n+1} & =w^{n} \cdot \boldsymbol{v}_{i}^{n}+c_{1} \cdot \operatorname{rand}(\boldsymbol{p} \boldsymbol{B e s t} \\
i
\end{array}-\boldsymbol{x}_{i}^{n}\right)+c_{2} \cdot \operatorname{rand}\left(\boldsymbol{g} \boldsymbol{B e s t}^{n}-\boldsymbol{x}_{i}^{n}\right), ~ \begin{aligned}
\boldsymbol{x}_{i}^{n+1} & =\boldsymbol{x}_{i}^{n}+\boldsymbol{v}_{i}^{n+1} \tag{10.7}
\end{aligned}
$$

where $i=1, \ldots, P ; c_{1}$ is known as the cognitive acceleration constant, which is the factor where the individual particle learns from itself; $c_{2}$ is known as the social acceleration constant, which is the factor where the individual particle learns from the society (swarm); both $c_{1}$ and $c_{2}$ are positive constants, which are usually $c_{1}=c_{2}=2$; rand is a random number in [0, 1]; and $v_{i}^{n+1}$ is the vector sum of $\boldsymbol{v}_{i}^{n}$, $\boldsymbol{\operatorname { B e s e t }} \boldsymbol{t}_{i}^{n}-\boldsymbol{x}_{i}^{n}$, and $\boldsymbol{g} \boldsymbol{B e s t} \boldsymbol{t}^{n}-\boldsymbol{x}_{i}^{n}$. Thus the combination diagram of the weighted values of the three possible movement directions of the particle is as shown in Figure 10.1.
In order to balance the global search ability and local optimization ability of particles, a particle in every dimension has a maximum velocity $V_{\max }^{l}$ set by the user ( $l$ is the dimensionality of the particle, where $1 \leq l \leq L$ and $V_{\max }^{l}>0$ ). If the upgraded velocity of a particle in a dimension exceeds $V_{\max }^{i}$, the velocity in this dimension is limited to $V_{\max }^{l}$; that is, if $\boldsymbol{v}_{i l}^{n+1}>V_{\max }^{l}$, then let $\boldsymbol{v}_{i l}^{n+1}=V_{\text {max }}^{l}$; if $v_{i l}^{n+1}<-V_{\text {max }}^{l}$, then let $\boldsymbol{v}_{i l}^{n+1}=-V_{\text {max }}^{l}$.
This parameter has proved to be very important to convergence of PSO in applications. If the selected $V_{\max }^{l}$ is excessively large, the particle may fly past the optimal solution; if the selected $V_{\max }^{l}$ is too small, the particle may be unable to detect the space outside the local optimum area, only being able to obtain the local optimal solution [6, 7].
The parameter $w$ is the inertia weight factor and was also introduced for the purpose of balancing the global search ability and local optimization ability of particles. It is usually linearly reduced from $w_{\text {max }}$ to $w_{\text {min }}$, so that the particle swarm algorithm has a better global search ability at the beginning of an iteration and higher local optimization accuracy can also be achieved in the later period of the iteration [6]. Selection of the inertia weight factor $w$ can


Figure 10.1 Combination of the weighted values of the three possible movement directions of the particle

Table 10.2 Calculation process of particle swarm optimization
Step 1. Randomly initialize the position and velocity of the particle in the solution space, calculate the fitness function value of particles (i.e., the corresponding function to be optimized), to obtain the globally optimal solution $\boldsymbol{g B}$ Best and the optimal solution $\boldsymbol{p B e s t} \boldsymbol{t}_{i}$ that the particle swarm has obtained until the current time (in the case of the minimization issue, the optimal solution is the one made the minimum value of the fitness function; otherwise, the maximum value)

Step 2. If the stopping condition of iteration has been reached, output the result and end the algorithm; otherwise, proceed to step 3

Step 3. Add the number of iteration(s) with one; calculate the inertia weighting factor at the time by expression (10.8), update the velocity component and position component of each particle using the expressions (10.6) and (10.7) (the velocity of a particle in a dimension should not exceed the maximum velocity in the dimension, but if the maximum velocity is exceeded, the maximum velocity $V_{\max }^{l}$ in this dimension is selected; likewise, the position of the particle should not be beyond the scope of the solution space). Calculate the fitness function of each particle, update the globally optimal solution $\boldsymbol{g} \boldsymbol{B e s t}$ and the optimal solution $\boldsymbol{p} \boldsymbol{B e s t} \boldsymbol{t}_{i}$ that the particle has obtained up until the current time; then proceed to step 2
be expressed as

$$
\begin{equation*}
w^{n}=w_{\max }-\frac{w_{\max }-w_{\min }}{\text { iter }_{\max }} n \tag{10.8}
\end{equation*}
$$

where $w_{\max }$ and $w_{\min }$ are respectively the maximum value and minimum value of the inertia weighting factor, iter $r_{\max }$ is the maximum number of iteration(s), and $n$ is the count number of the current iteration. The initial position and velocity of every particle is randomly generated in the solution space and then iteration calculation is performed using expressions (10.6) to (10.8), until a satisfactory solution is obtained or the maximum number of iteration(s) has been reached. The calculation process of the PSO algorithm is shown in Table 10.2.

### 10.2.2 Orbit Determination Method Based on the PSO Algorithm

When making an estimation of the two state parameters under the case of the unknown target signal carrier frequency $f_{0}$ and known $f_{0}$, it should be realized that what they have in common in two cases is that they both need estimation of the orbit elements $\boldsymbol{X}_{F O}$ of the satellite at the epoch time. The difference between them is that, in the case of the unknown carrier frequency $f_{0}$, the value of $f_{0}$ also needs to be estimated in addition to estimation of $\boldsymbol{X}_{F O}$. The two cases are analyzed in the following:

The first case refers to the fact that there is no prior information for the noncooperative target. In this case, the target signal carrier frequency $f_{0}$ is unknown and the search scope for the state parameter to be estimated in each dimension is relatively large. Specifically, the semi-major axis $a$ can be set in a large search scope according to the orbital altitude of the target of interest; the value range of eccentricity $e$ is $[0,1)$; the value range of orbit inclination $i$ is $[0, \pi)$; and the value range of the remaining three parameters to be estimated is $[0,2 \pi)$.
The second case refers to the fact that there is certain prior information for the noncooperative target; for example, the rough information on the target orbit has been obtained in other
ways. In this case, the assumption is that the satellite signal carrier frequency $f_{0}$ is known and that the search scope of the state parameters in each dimension is relatively smaller.

The orbit (state) parameter estimation method based on PSO can be expressed as follows:

1. Randomly generate the position and velocity of each particle in the search space, with the assumption that the number of particles in PSO is $P$.
2. Regard each particle as a set of feasible solutions, that is, the orbit elements of the satellite at the epoch time, and convert them to the position and velocity vector at the corresponding $N$ observation times using the orbit extrapolation method previously applied in the case of two-body model and with consideration of $\mathrm{J}_{2}$ perturbation of earth oblateness. Then substitute the position and velocity vector of the observing satellite at the corresponding time into the expression (10.3). For the unknown $f_{0}$, the least squares method can be employed to estimate $f_{0}$, which is expressed as $\widehat{f}_{0}=\left(\boldsymbol{H}_{i}^{T} \boldsymbol{H}_{i}\right)^{-1} \boldsymbol{H}_{i}^{T} \boldsymbol{Z}_{\mathbf{m}}$, where $\boldsymbol{H}_{i}=\boldsymbol{H}_{F O}^{i}\left(\boldsymbol{X}_{F O}^{i}\right)$ is the result of solving the $i$ th particle.
3. Substitute the known carrier frequency $f_{0}$ or the least squares estimate of the carrier frequency $\widehat{f}_{0}$ (in the case of $f_{0}$ unknown) into the expression (10.3). Then the state parameter to be estimated, $\boldsymbol{X}_{F O}$, can be obtained according to the expression

$$
\begin{equation*}
\widehat{\boldsymbol{X}}_{F O}=\underset{\boldsymbol{X}_{F O}^{i}}{\arg \min } J\left(\boldsymbol{X}_{F O}^{i}\right), \tag{10.9}
\end{equation*}
$$

where $J\left(\boldsymbol{X}_{F O}^{i}\right)=\left\|\boldsymbol{Z}_{\mathrm{m}}-\bar{f}_{0} \cdot \boldsymbol{H}_{F O}\left(\boldsymbol{X}_{F O}^{i}\right)\right\|^{2}$, which is the cost function to be optimized in the PSO algorithm, that is, the fitness function (if $f_{0}$ is known, $\bar{f}_{0}=f_{0}$; if $f_{0}$ is unknown, $\bar{f}_{0}=\widehat{f}_{0}$ ). Then repeat step 3 of the PSO algorithm until the stopping condition has been reached.

### 10.3 System Observability Analysis

Chapter 8 analyzed the existing methods for an observability description and the problems of such methods. It also introduced a new method for a description of system observability and explained the advantages of that new method for a description of system observability. In addition, for a satellite-to-satellite passive orbit determination system based on BO and on bearings and frequency information, Chapters 8 and 9 also researched the correlation between the system observability and the influencing factors for the state equation and the measurement equation under typical simulation scenarios.
In the satellite-to-satellite passive orbit determination system based on BO and on bearings and frequency information, the position and velocity vector in the J2000.0 coordinate system are used as state variables. The position and velocity vector and the orbit elements of the satellite can be converted to each other, but the former changes with time. Thus, the position and velocity vector in a Cartesian coordinates system is not suitable for analysis of the effects on system observability of satellite orbit elements.
This chapter uses the orbit elements of the target satellite at the epoch time as the state parameters to be estimated, which, compared with the Cartesian coordinates system, is more helpful for analysis of the effects on orbit elements of system observability, that is, the effects of different geometrical types of the satellite on system observability. This research has the following two features:

1. Theory and analysis on this problem faces great difficulties. Both the observer and the target are satellites with six orbit elements needed to determine the orbit of the satellite; therefore the relative observation geometry will be jointly determined by 12 different parameters. It is rather difficult to find an analytic method to derive the effects of different parameter values on system observability.
2. Research into this problem has great significance in practice. If the result of the research into this problem can be obtained, system observability under the given satellite observation geometric conditions can be qualitatively analyzed and better results on the orbit of the target can be estimated through orbit maneuver of the observing satellite.

Due to the particular importance of this problem, it is researched in this chapter. However, during the research, the problem was simplified, which is mainly reflected in the following:

1. The research probed into the relation between the system observability and the orbit elements of the target satellite under specific observation geometric conditions.
2. The researches only focused on the effects of the first four orbit elements of the target satellite. Of the classic orbit elements of the satellite, the semi-major axis $a$ and eccentricity $e$ determine the size and shape of the elliptic orbit of the satellite, the orbit inclination $i$ and RAAN $\Omega$ determine the position of the orbital plane in 3D space, the argument of perigee $\omega$ determines the bearing of the ellipse in the orbital plane, and $M_{0}$ determines the mean argument of perigee of the satellite at time $t=0$. The first four orbit elements determine the position in the space of the satellite orbit and the size and shape of the elliptic orbit while the last two orbit elements only determine the position of the perigee in the orbital plane and the starting time of satellite motion. In other words, it is mainly the first four orbit elements that determine the different observation geometric relations between two satellites.

Intuitionally speaking, no matter what kind of observation information is applied, as long as the relative movement between two satellites intensifies, better state estimation results can be obtained with the same measurement accuracy; conversely, if the same state estimation accuracy can be achieved with lower requirements for the parameter measurement accuracy, this indicates that the system observability has been enhanced.
With specific orbital altitude of the observing satellite, the orbit period of the target satellite will increase as its semi-major axis increases and the relative movement between satellites will decrease accordingly; in other words, system observability reduces as the semi-major axis of the satellite increases.
By the 'Access' function used in $\mathrm{STK}^{\circledR}{ }^{\circledR} 6.0$, the following can be obtained through different scenario settings: for two satellites operating in different orbits, a longer observation arc can be acquired in the case where their orbital planes are almost perpendicular than in the case where they are almost in the same plane. Take the angle measurement information as the example. If two satellites not on the same track are almost on the same plane, the issue downgrades to a two-dimensional tracking issue and now the elevation angle has lost its importance, that is, the target orbit is determined by only the azimuth angle. However, in the case of perpendicular planes, the bearing measurement information and elevation measurement information can be used for comparison and, with the same measurement accuracy, the perpendicular planes case can result in higher estimation accuracy. In other words, the relative movement is much more significant in the case where the two satellite orbital planes are perpendicular than in the case

Table 10.3 Orbit elements of satellites in the analysis on system observability

| Description of orbit | Observing <br> elements <br> Satellite of <br> scenalation 1 | Observing <br> satellite of <br> Simulation <br> scenario 2 | Observing <br> satellite of <br> Simulation <br> scenario 2 | Default <br> scenario of <br> target satellite <br> $(T)$ |
| :--- | :---: | :---: | :---: | :---: |
| Semi-major axis $a(\mathrm{~km})$ | 42000 | 40000 | 28000 | 7200 |
| Eccentricity $e$ | 0.1 | 0.01 | 0 | 0.05 |
| Orbit inclination $i(\mathrm{deg})$ | 36 | 45 | 120 | 30 |
| RAAN $\Omega($ deg $)$ | 120 | 30 | 60 | 75 |
| Argument of perigee $\omega(\mathrm{deg})$ | 60 | 30 | 30 | 60 |
| Mean argument of perigee $M_{0}(\mathrm{deg})$ | 0 | 0 | 0 | 10 |

where the two satellite orbital planes are almost on the same plane. In summary, compared with two satellites not on the same track but almost on the same plane, two satellites being almost perpendicular provides a longer observation duration and also improves the estimation performance.
The analysis above explains the effects of different observation geometric conditions on system observability. The following will, through computer simulation, analyze the relation between the system observation and the first four orbit elements of the satellite under a specific observation geometric condition.
Assuming that the filtering period is $T=20$ seconds and the observation duration is 9000 seconds (the simulation was intended for a LEO (low earth orbit) satellite with orbital altitude $<3000 \mathrm{~km}$ and orbital period $<9000$ seconds, thus making the observation duration 9000 seconds here, so that the operation of the LEO satellite in one entire orbital period can be learned), the Doppler frequency measurement error is $\sigma_{f}=50 \mathrm{~Hz}$ and the signal carrier frequency of the target satellite is set as $f_{0}=15 \mathrm{GHz}$. In three different simulation scenarios, the orbit elements of the observing satellite and the default orbit elements of the target satellite are as shown in Table 10.3.

### 10.3.1 Simulation Scenario 1

From the first set of orbit elements of the observing satellite in Table 10.3, the curve of the relation between system observability and the semi-major axis $a_{2}$ and eccentricity $e_{2}$ can be obtained, which is shown in Figure 10.2 (define the $\sigma_{1}$ mark as the orbit elements of the observing satellite and the $\sigma_{2}$ mark as the orbit elements of the target satellite, where $\sigma=\left[\begin{array}{lll}a & e & i\end{array}\right]$ ). The relation between system observability and the orbit inclination $i_{2}$ and RAAN $\Omega_{2}$ is indicated in the curve in Figure 10.3.
Figure 10.2 indicates that the value of system observability increases as the semi-major axis $a_{2}$ increases. When eccentricity $e_{2}$ is relatively small, the observability is fairly stable; as eccentricity $e_{2}$ increases, system observability changes in a wide range. However, in general, when $e_{2}=0.05$, the value of observability is small and stable. Figure 10.3 indicates that when the orbit inclination $i_{2}$ is close to or equal to the orbit inclination of the observing satellite (in this scenario, the orbit inclination of the observing satellite is $i_{1}=36^{\circ}$ ), the value of system


Figure 10.2 Relation between observability and the semi-major axis $a_{2}$ and eccentricity $e_{2}$ in Scenario 1. (a) Degree of observability of $a_{2}$ and (b) degree of observability of $e_{2}$
observability is larger; as the orbit inclination $i_{2}$ is different from the observing satellite orbit inclination $i_{1}$, the value of system observability reduces to a certain extent; and when $i_{2}=120^{\circ}$, the value of system observability is obviously reduced. When the RAAN $\Omega_{2}$ is close to or equal to the RAAN of the observing satellite (in this scenario, the RAAN of the observing satellite is $\Omega_{1}=120^{\circ}$ ), the value of system observability is larger; as the RAAN $\Omega_{2}$ moves far away from the RAAN of the observing satellite $\Omega_{1}$, the value of system observability reduces to a certain extent; and when $\Omega_{2}=30^{\circ}$, the value of system observability is obviously reduced.

### 10.3.2 Simulation Scenario 2

By using the second set of orbit elements of the observing satellite in Table 10.3, the curve of the relation between system observability and the semi-major axis $a_{2}$ and eccentricity $e_{2}$ can


Figure 10.3 Relation between observability and orbit inclination $i_{2}$ and RAAN $\Omega_{2}$. (a) Degree of observability of $i_{2}$ and (b) degree of observability of $\Omega_{2}$
be obtained, which is shown in Figure 10.4. The relation between system observability and the orbit inclination $i_{2}$ and the RAAN $\Omega_{2}$ is shown in the curve in Figure 10.5.
Figure 10.4 indicates that the value of system observability increases as the semi-major axis $a_{2}$ increases. When eccentricity $e_{2}$ is relatively small, the observability is fairly stable; as eccentricity $e_{2}$ increases, system observability changes in a wide range; but, in general, when $e_{2}=0.05$, the value of observability is small and stable. Figure 10.5 indicates that when the orbit inclination $i_{2}$ is close to or equal to the orbit inclination of the observing satellite (in this scenario, the orbit inclination of the observing satellite is $i_{1}=45^{\circ}$ ), the value of system observability is larger, as the value of the orbit inclination $i_{2}$ moves far away from the observing satellite orbit inclination $i_{1}$, the value of system observability reduces to a certain extent; and


Figure 10.4 Relation between observability and the semi-major axis $a_{2}$ and eccentricity $e_{2}$ in Scenario 2. (a) Degree of observability of $a_{2}$ and (b) degree of observability of $e_{2}$
when $i_{2}=135^{\circ}$, the value of system observability is obviously reduced. When the RAAN $\Omega_{2}$ is close to or equal to RAAN of the observing satellite (in this scenario, the RAAN of the observing satellite is $\Omega_{1}=30^{\circ}$ ), the value of system observability is larger; as the RAAN $\Omega_{2}$ moves far away from the RAAN of the observing satellite $\Omega_{1}$, the value of system observability reduces to certain extent; and when $\Omega_{2}=120^{\circ}$, the value of system observability is obviously reduced.

### 10.3.3 Simulation Scenario 3

From using the third set of orbit elements of the observing satellite in Table 10.3, the curve of the relation between system observability and the semi-major axis $a_{2}$ and the eccentricity $e_{2}$


Figure 10.5 Relation between observability and the orbit inclination $i_{2}$ and RAAN $\Omega_{2}$ in Scenario 2 . (a) Degree of observability of $i_{2}$ and (b) degree of observability of $\Omega_{2}$
can be obtained, as shown in Figure 10.6. The relation between system observability and the orbit inclination $i_{2}$ and the RAAN $\Omega_{2}$ is indicated in the curve in Figure 10.7.
Figure 10.6 indicates that the value of system observability increases as the semi-major axis $a_{2}$ increases. When the eccentricity $e_{2}$ is relatively small, the observability is fairly stable; as the eccentricity $e_{2}$ increases, system observability changes dramatically; but, in general, when $e_{2}=0.05$, the value of observability is small and stable. Figure 10.7 indicates that when the orbit inclination $i_{2}$ is close to or equal to the orbit inclination of the observing satellite (in this scenario, the orbit inclination of the observing satellite is $i_{1}=120^{\circ}$ ), the value of system observability is larger; as the orbit inclination $i_{2}$ is different from the observing satellite orbit inclination $i_{1}$, the value of system observability reduces to a certain level; and when $i_{2}=30^{\circ}$, the value of system observability is obviously reduced. When the RAAN $\Omega_{2}$ is close to or equal


Figure 10.6 Relation between observability and the semi-major axis $a_{2}$ and eccentricity $e_{2}$ in Scenario 3. (a) Degree of observability of $a_{2}$ and (b) degree of observability of $e_{2}$
to the RAAN of the observing satellite (in this scenario, the RAAN of the observing satellite is $\Omega_{1}=60^{\circ}$ ), the value of system observability is larger; as the RAAN $\Omega_{2}$ moves far away from the RAAN of the observing satellite $\Omega_{1}$, the value of system observability reduces to a certain extent; and when $\Omega_{2}=150^{\circ}$, the value of system observability is obviously reduced.
In conclusion, through the three sets of different observing satellites above, the relation between the observability of the satellite-to-satellite passive orbit determination system with frequency-only measurement and the orbit elements of the target satellite was analyzed. Under the observation condition of a single HEO (high earth orbit) satellite to locate the LEO satellite target, the following conclusions are acquired by simulation:

1. The system observability reduces as the semi-major axis $a_{2}$ of the target satellite increases.
2. The system observability increases first and then reduces as the eccentricity $e_{2}$ of the target satellite increases; when $0.001<e_{2}<0.1$, the observability is favorable.
3. When the orbit inclination $i_{2}$ of the target satellite is close to or equal to the orbit inclination $i_{1}$ of the observing satellite, the observability downgrades significantly; when there is a certain difference between the value of the orbit inclination $i_{2}$ of the target satellite and $i_{1}$ of the observing satellite, the observability improves accordingly; and if the difference between the value of the orbit inclination $i_{2}$ of the target satellite and $i_{1}$ of the observing satellite is $90^{\circ}$ or almost $90^{\circ}$, the system observability improves significantly.
4. When the RAAN $\Omega_{2}$ of the target satellite is close to or equal to the RAAN $\Omega_{1}$ of the observing satellite, observability significantly downgrades; when there is a certain difference between the value of the RAAN $\Omega_{2}$ of the target satellite and the RAAN $\Omega_{1}$ of the observing satellite, the observability improves accordingly; and if the difference between the value of the RAAN $\Omega_{2}$ of the target satellite and the value of the RAAN $\Omega_{1}$ of the observing satellite is $90^{\circ}$ or almost $90^{\circ}$, the system observability improves significantly.


Figure 10.7 Relation between observability and the orbit inclination $i_{2}$ and RAAN $\Omega_{2}$ in Scenario 3 . (a) Degree of observability of $i_{2}$ and (b) degree of observability of $\Omega_{2}$

The conclusions acquired above are based on the method of passive orbit determination with frequency-only measurement, but the relation between the system observability and the orbit elements of the target satellite can be expanded to the corresponding cases where other observed quantities are used, although the magnitude and variation trend of the system observability acquired are not completely the same.

### 10.4 CRLB of the Orbit Parameter Estimation Error

For cases of either unknown or known signal carrier frequencies $f_{0}$, because the state parameters to be estimated are orbit elements of the target satellite at the epoch time (when the signal carrier frequency $f_{0}$ is unknown, $f_{0}$ becomes a parameter to be additionally estimated), the CRLB (Cramér-Rao lower bound) analysis of the orbit parameter estimation error discussed below will mainly aim at the case of unknown $f_{0}$. The calculation expressions of the parameter estimation error CRLB for the case of known $f_{0}$ are the same as those of the first six parameters for the case of unknown $f_{0}$.

The parameters to be estimated for the case of unknown $f_{0}$ can be expressed as

$$
\begin{equation*}
\boldsymbol{\theta}=\left[\theta_{1}, \theta_{2}, \ldots, \theta_{7}\right]=\left[\boldsymbol{X}_{F O}^{\boldsymbol{T}}, f_{0}\right] . \tag{10.10}
\end{equation*}
$$

According to expression (10.3), suppose errors $\varepsilon_{1}, \ldots, \varepsilon_{N}$ are zero-mean Gaussian and i.i.d. with variance $\sigma_{f}^{2}$, that is, $\varepsilon_{i} \sim N\left(0, \sigma_{f}^{2}\right)(1 \leq i \leq N)$. Therefore, for a given $\theta$, the joint conditional probability density function (PDF) of $f_{m}^{1}, \ldots, f_{m}^{N}$ can be expressed as

$$
\begin{equation*}
p\left(f_{m}^{1}, \ldots, f_{m}^{N} \mid \boldsymbol{\theta}\right)=\frac{1}{\left(2 \pi \sigma_{f}^{2}\right)^{N / 2}} \exp \left[-\frac{\sum_{i=1}^{N}\left(\bar{f}_{m}^{i}\right)^{2}}{2 \sigma_{f}^{2}}\right] \tag{10.11}
\end{equation*}
$$

where $\bar{f}_{m}^{i}$ is

$$
\begin{equation*}
\bar{f}_{m}^{i}=E\left(f_{m}^{i}\right)=f_{0}\left[1-\frac{\left(\dot{\boldsymbol{r}}_{i} \cdot \boldsymbol{r}_{i}\right)}{c \cdot r_{i}}\right] \tag{10.12}
\end{equation*}
$$

where $E\{\cdot\}$ represents a mathematical expectation operation. By taking the natural logarithm of expression (10.11), it can be found that

$$
\begin{equation*}
\ln (p \mid \boldsymbol{\theta})=-\frac{\sum_{i=1}^{N}\left(f_{m}^{i}-\bar{f}_{m}^{i}\right)^{2}}{2 \sigma_{f}^{2}}-\frac{N}{2} \ln \left(2 \pi \sigma_{f}^{2}\right) \tag{10.13}
\end{equation*}
$$

Then by taking the partial differentials of expression (10.13) with respect to each $\theta$ component, it can be found that

$$
\begin{equation*}
\frac{\partial \ln (p \mid \theta)}{\partial \theta_{j}}=\frac{\sum_{i=1}^{N}\left(f_{m}^{i}-\bar{f}_{m}^{i}\right)}{\sigma_{f}^{2}} \frac{\partial \bar{f}_{m}^{i}}{\partial \theta_{j}}, \tag{10.14}
\end{equation*}
$$

that is, the $j k$ th element of the Fisher information matrix (FIM) is expressed as

$$
\begin{equation*}
\boldsymbol{F}_{j k}=E\left\{\frac{\sum_{i=1}^{N}\left(f_{m}^{i}-\bar{f}_{m}^{i}\right)}{\sigma_{f}^{2}} \frac{\partial \bar{f}_{m}^{i}}{\partial \theta_{j}} \frac{\sum_{q=1}^{N}\left(f_{m}^{q}-\bar{f}_{m}^{q}\right)}{\sigma_{f}^{2}} \frac{\partial \bar{f}_{m}^{q}}{\partial \theta_{k}}\right\}=\frac{1}{\sigma_{f}^{2}} \sum_{i=1}^{N} \frac{\partial \bar{f}_{m}^{i}}{\partial \theta_{j}} \frac{\partial \bar{f}_{m}^{i}}{\partial \theta_{k}} \tag{10.15}
\end{equation*}
$$

According to expression (10.15), it can be found that the CRLB of every parameter $\theta_{j}$ in the vector $\boldsymbol{\theta}$ to be estimated is $\left(\boldsymbol{F}^{-1}\right)_{j j}$.
Since $\bar{f}_{m}^{i}$ is a function of $\theta_{j}(1 \leq j \leq 7)$, when finding $\partial \bar{f}_{m}^{i} / \partial \theta_{j}$ from the FIM, first find the partial differentials of the frequency component $\bar{f}_{m}^{i}$ with respect to the location and velocity vectors of target satellite and then find the partial differentials of the location and velocity vectors of target satellite with respect to $\theta_{j}$. Finally, multiple each calculation result above to obtain $\partial \bar{f}_{m}^{i} / \partial \theta_{j}$. See the following for detailed solutions.
According to expression (10.12), it can be found that $\partial \bar{f}_{m}^{i} / \partial \theta_{j}(1 \leq j \leq 7)$ can be expressed as follows:

$$
\begin{equation*}
\frac{\partial \bar{f}_{m}^{i}}{\partial \theta_{j}}=\frac{\partial \bar{f}_{m}^{i}}{\partial \boldsymbol{r}^{T}} \frac{\partial \boldsymbol{r}}{\partial \theta_{j}}+\frac{\partial \bar{f}_{m}^{i}}{\partial \dot{\boldsymbol{r}}^{T}} \frac{\partial \dot{\boldsymbol{r}}}{\partial \theta_{j}}+\frac{\partial \bar{f}_{m}^{i}}{\partial f_{0}} \frac{\partial f_{0}}{\partial \theta_{j}}, \tag{10.16}
\end{equation*}
$$

where $\boldsymbol{r}$ and $\boldsymbol{r}$ are the relative location and the relative velocity vectors respectively. According to expression (10.12), it can be found that $\partial \bar{f}_{m}^{i} / \partial \mathbf{r}^{\mathrm{T}}, \partial \bar{f}_{m}^{i} / \partial \mathrm{r}^{\mathrm{T}}$, and $\partial \bar{f}_{m}^{i} / \partial f_{0} m$ can be expressed as

$$
\begin{align*}
& \frac{\partial \bar{f}_{m}^{i}}{\partial \boldsymbol{r}^{\mathrm{T}}}=\frac{f_{0}}{c}\left(\frac{(\dot{\boldsymbol{r}} \cdot \boldsymbol{r}) \boldsymbol{r}^{\mathrm{T}}-r^{2} \dot{\boldsymbol{r}}^{\mathrm{T}}}{r^{3}}\right)  \tag{10.17}\\
& \frac{\partial \bar{f}_{m}^{i}}{\partial \dot{\boldsymbol{r}}^{\mathrm{T}}}=-\frac{f_{0}}{c}\left(\frac{\dot{\boldsymbol{r}}^{\mathrm{T}}}{r}\right)  \tag{10.18}\\
& \frac{\partial \bar{f}_{m}^{i}}{\partial f_{0}}=1-\frac{(\dot{\boldsymbol{r}} \cdot \boldsymbol{r})}{c r} \tag{10.19}
\end{align*}
$$

According to definitions of the relative location and relative velocity vectors, it can be found that $\partial r / \partial \theta_{j}$ and $\partial \dot{r} / \partial \theta_{j}$ are equal to $\partial \boldsymbol{r}_{T} / \partial \theta_{j}$ and $\partial \dot{r}_{T} / \partial \theta_{j}$, respectively, and the solutions for $\partial \boldsymbol{r}_{T} / \partial \theta_{j}$ and $\partial \dot{r}_{T} / \partial \theta_{j}(1 \leq j \leq 7)$ can be found as follows.
According to the Appendix, in the case of the two-body model, the transformation relations between the orbit elements of the target satellites and their location and velocity vectors can be expressed in the form of vectors as follows:

$$
\begin{align*}
& \boldsymbol{r}_{T}=a \boldsymbol{A} \cdot\left(\begin{array}{c}
\cos E-e \\
\sqrt{1-e^{2}} \sin E \\
0
\end{array}\right)=a \boldsymbol{A} \cdot \boldsymbol{V}_{r},  \tag{10.20}\\
& \dot{\boldsymbol{r}}_{T}=\frac{\sqrt{\mu / a}}{(1-e \cos E)} \boldsymbol{A} \cdot\left(\begin{array}{c}
-\sin E \\
\sqrt{1-e^{2}} \cos E \\
0
\end{array}\right)=\frac{\sqrt{\mu / a}}{(1-e \cos E)} \boldsymbol{A} \cdot \boldsymbol{V}_{\dot{r}}, \tag{10.21}
\end{align*}
$$

where $\mu$ is the gravitational constant. For the convenience of expression, use $\boldsymbol{V}_{r}$ and $\boldsymbol{V}_{\dot{r}}$ to substitute the corresponding vectors and $E$ to express the eccentric argument of perigee, which is the nonlinear function of $a, e$, and $M_{0}$ [12] and can be expressed as follows:

$$
\begin{equation*}
E-e \sin E=\sqrt{\mu / a^{3}} t+M_{0} \tag{10.22}
\end{equation*}
$$

$\boldsymbol{A}$ is a transition matrix defined as follows:

$$
\boldsymbol{A}=\left(\begin{array}{lll}
A_{11} & A_{12} & A_{13}  \tag{10.23}\\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{array}\right)
$$

Each element of matrix $\boldsymbol{A}$ can be expressed respectively as follows:

$$
\left.\begin{array}{l}
A_{11}=\cos \Omega \cos \omega-\sin \Omega \sin \omega \cos i  \tag{10.24}\\
A_{12}=-\cos \Omega \sin \omega-\sin \Omega \cos \omega \cos i \\
A_{13}=\sin \Omega \sin i \\
A_{21}=\sin \Omega \cos \omega+\cos \Omega \sin \omega \cos i \\
A_{22}=-\sin \Omega \sin \omega+\cos \Omega \cos \omega \cos i . \\
A_{23}=-\cos \Omega \sin i \\
A_{31}=\sin i \sin \omega \\
A_{32}=\sin i \cos \omega \\
A_{33}=\cos i
\end{array}\right\}
$$

According to expression (10.22), $\partial E / \partial a, \partial E / \partial e$, and $\partial E / \partial M_{0}$ can be respectively expressed as:

$$
\left.\begin{array}{l}
\frac{\partial E}{\partial a}=\frac{-\frac{3}{2} \sqrt{\mu / a^{5}} t}{1-e \cos E}  \tag{10.25}\\
\frac{\partial E}{\partial e}=\frac{\sin E}{1-e \cos E} \\
\frac{\partial E}{\partial M_{0}}=\frac{1}{1-e \cos E}
\end{array}\right\}
$$

Then each component of $\partial r_{T} / \partial \theta_{j}$ and $\partial \dot{r}_{T} / \partial \theta_{j}(1 \leq j \leq 6)$ can be calculated according to expressions (10.20) to (10.25) and $\partial r_{T} / \partial a$ and $\partial \dot{r}_{T} / \partial a$ can be expressed respectively as follows:

$$
\left.\begin{array}{c}
\frac{\partial \boldsymbol{r}_{T}}{\partial a}=\boldsymbol{A} \cdot\binom{\cos E-e-a \sin E \frac{\partial E}{\partial a}}{\sqrt{1-e^{2}} \sin E+a \sqrt{1-e^{2}} \cos E \frac{\partial E}{\partial a}} \\
0
\end{array}\right] \begin{gathered}
-\sqrt{a} \frac{\partial E}{\partial a}(\cos E-e)+\sin E \frac{0.5}{\sqrt{a}}(1-e \cos E)  \tag{10.27}\\
\frac{\partial \dot{\boldsymbol{r}}_{T}}{\partial a}=\frac{\sqrt{\mu}}{a(1-e \cos E)^{2}} \boldsymbol{A} \cdot\binom{-\sqrt{1-e^{2}}\left(\sqrt{a} \sin E \frac{\partial E}{\partial a}+\frac{0.5}{\sqrt{a}} \cos E(1-e \cos E)\right.}{0}
\end{gathered}
$$

and $\partial r_{T} / \partial e$ and $\partial \dot{r}_{T} / \partial e$ can be expressed respectively as follows:

$$
\begin{gather*}
\frac{\partial \boldsymbol{r}_{T}}{\partial e}=a \boldsymbol{A} \cdot\left(\begin{array}{c}
-\sin E \frac{\partial E}{\partial e}-1 \\
\frac{-e}{\sqrt{1-e^{2}}} \sin E+\sqrt{1-e^{2}} \cos E \frac{\partial E}{\partial e} \\
0
\end{array}\right)  \tag{10.28}\\
\frac{\partial \dot{\boldsymbol{r}}_{T}}{\partial e}=\frac{\sqrt{\mu}}{a(1-e \cos E)^{2}} \boldsymbol{A} \cdot\left(\begin{array}{c}
-\sqrt{a} \frac{\partial E}{\partial e}(\cos E-e)-\sqrt{a} \sin E \cos E \\
\frac{\sqrt{a} \cos E}{\sqrt{1-e^{2}}}(\cos E-e)-\sqrt{a} \sqrt{1-e^{2}} \sin E \frac{\partial E}{\partial e} \\
0
\end{array}\right) \tag{10.29}
\end{gather*}
$$

and $\frac{\partial \boldsymbol{r}_{T}}{\partial i}$ and $\frac{\partial \dot{\boldsymbol{r}}_{T}}{\partial i}$ can be expressed respectively as follows:

$$
\begin{align*}
\frac{\partial \boldsymbol{r}_{T}}{\partial i} & =a\left(\begin{array}{ccc}
\sin \Omega \sin \omega \sin i & \sin \Omega \cos \omega \sin i & \sin \Omega \cos i \\
-\cos \Omega \sin \omega \sin i & -\cos \Omega \cos \omega \sin i & -\cos \Omega \cos i \\
\cos i \sin \omega & \cos i \cos \omega & -\sin i
\end{array}\right) \cdot \boldsymbol{V}_{r},  \tag{10.30}\\
\frac{\partial \dot{\boldsymbol{r}}_{T}}{\partial i} & =\frac{\sqrt{\mu}}{a(1-e \cos E)}\left(\begin{array}{ccc}
\sin \Omega \sin \omega \sin i & \sin \Omega \cos \omega \sin i & \sin \Omega \cos i \\
-\cos \Omega \sin \omega \sin i & -\cos \Omega \cos \omega \sin i & -\cos \Omega \cos i \\
\cos i \sin \omega & \cos i \cos \omega & -\sin i
\end{array}\right) \cdot \boldsymbol{V}_{\dot{r}} . \tag{10.31}
\end{align*}
$$

and $\partial \boldsymbol{r}_{T} / \partial \Omega$ and $\partial \dot{r}_{T} / \partial \Omega$ can be expressed respectively as follows:

$$
\begin{align*}
& \frac{\partial \boldsymbol{r}_{T}}{\partial \Omega}=a \cdot \boldsymbol{A}_{\Omega} \cdot \boldsymbol{V}_{r}  \tag{10.32}\\
& \frac{\partial \dot{\boldsymbol{r}}_{T}}{\partial \Omega}=\frac{\sqrt{\mu}}{a(1-e \cos E)} \cdot \boldsymbol{A}_{\Omega} \cdot \boldsymbol{V}_{\dot{r}} \tag{10.33}
\end{align*}
$$

and finally $\partial \boldsymbol{r}_{T} / \partial \omega$ and $\partial \dot{r}_{T} / \partial \omega$ can be expressed respectively as follows:

$$
\begin{align*}
& \frac{\partial \boldsymbol{r}_{T}}{\partial \omega}=a \cdot \boldsymbol{A}_{\omega} \cdot \boldsymbol{V}_{r}  \tag{10.34}\\
& \frac{\partial \dot{\boldsymbol{r}}_{T}}{\partial \omega}=\frac{\sqrt{\mu}}{a(1-e \cos E)} \cdot \boldsymbol{A}_{\omega} \cdot \boldsymbol{V}_{\dot{r}} \tag{10.35}
\end{align*}
$$

where $\boldsymbol{A}_{\Omega}$ and $\boldsymbol{A}_{\boldsymbol{\omega}}$ represent the partial differentials of $\Omega$ and $\omega$ in matrix $\boldsymbol{A}$ of expression (10.23), respectively, and $\partial \boldsymbol{r}_{T} / \partial M_{0}$ and $\partial \boldsymbol{r}_{T} / \partial M_{0}$ can be expressed respectively as follows:

$$
\frac{\partial \boldsymbol{r}_{T}}{\partial M_{0}}=a \boldsymbol{A} \cdot\left(\begin{array}{c}
-\sin E \frac{\partial E}{\partial M_{0}}  \tag{10.36}\\
\sqrt{1-e^{2}} \cos E \frac{\partial E}{\partial M_{0}} \\
0
\end{array}\right)
$$

$$
\frac{\partial \dot{\boldsymbol{r}}_{T}}{\partial M_{0}}=\frac{\sqrt{\mu}}{a(1-e \cos E)^{2}} \boldsymbol{A} \cdot\left(\begin{array}{c}
-\sqrt{a} \frac{\partial E}{\partial M_{0}}(\cos E-e)  \tag{10.37}\\
-\sqrt{a} \sqrt{1-e^{2}} \sin E \frac{\partial E}{\partial M_{0}} \\
0
\end{array}\right)
$$

In addition, there is

$$
\frac{\partial \boldsymbol{r}_{T}}{\partial f_{0}}=\frac{\partial \dot{\boldsymbol{r}}_{T}}{\partial f_{0}}=\frac{\partial f_{0}}{\partial \theta_{j}}=0(1 \leq j \leq 6)
$$

Substitute the solution results of the partial differentials above into expression (10.15) and the estimation error lower bond (CRLB) of the parameters can be obtained.

### 10.5 Orbit Determination and Tracking Simulation and Its Analysis

In this section, a performance simulation analysis of the frequency-only based passive orbit determination described in Section 10.2 will be simulated under different conditions. The system modus is still established for two cases: one for the two-body model and the other for considering $J_{2}$ perturbation of earth oblateness.
Generally speaking, factors influencing the nonlinear estimator performance include various parameters, such as the initial state error (ISE), the target model error, a number of accumulated measuring points, and so on. In this section, the performance of frequency-only based passive orbit determination from effects such as the ISE, the frequency measurement error, accumulated measuring points (equal to the length of observation arcs here), observation intervals is analyzed for the case of the target signal carrier frequency $f_{0}$, known or not known, to discover the influences of different factors on estimation accuracy. The corresponding conclusions are given.
Suppose that the frequency and angle accumulated measuring points of the target satellite (T) measured by the observing satellite ( O ) in every $T_{s}$ seconds are $N_{f}$ and $N_{\beta}$, respectively, the corresponding tracked durations are $N_{f} T_{s}$ and $N_{\beta} T_{s}$, the angle measurement error, frequency measurement error, and the signal carrier frequencies of the target satellite are $\sigma_{\beta}$ and $\sigma_{\varepsilon}$ (suppose $\sigma_{\beta}=\sigma_{\varepsilon}$ ), $\sigma_{f}$ and $f_{0}=15 \mathrm{GHz}$, respectively.
According to the analysis of Section 10.2.2 in this chapter, for known $f_{0}$ cases, the prior information of targets indicates that its parameter search scope is relatively small so its measurement accuracy is higher than that in the case of unknown $f_{0}$ in the simulation. The corresponding parameters used in the simulation scenarios are shown in Table 10.4. Orbit elements of the two satellites are shown in Table 10.5.
Corresponding parameters for the PSO algorithm are as follows: the number of swarm particles and the maximum iteration times iter $_{\max }$ of each dimension, which are 80 and 200, respectively. Two acceleration constants $c_{1}=c_{2}=2$, and the maximum and minimum inertia weight factors $w_{\text {max }}$ and $w_{\text {min }}$ and the maximum velocity $V_{\text {max }}^{l}(l$ is the dimension of particles and $1 \leq l \leq L$ ) are $1.2,0.2$, and $15 \%$ of the search scope of the dimension, respectively. For both the unknown signal carrier frequency $f_{0}$ case and the known $f_{0}$ case, the parameter search scopes of the PSO algorithm are shown in Table 10.6. The ISEs of the BO method are given in reference [1].

Table 10.4 Parameters in the frequency-only based passive orbit determination simulation

| Parameter description | Unknown $f_{0}$ |  |  | Known $f_{0}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T_{s}(\mathrm{~s})$ | 20 |  |  | 10 |  |  |
| Accumulated measuring points $N_{f}$ $\sigma_{f}(\mathrm{~Hz})$ | Case 1 | Case 2 | Case 3 | Case 1 | Case 2 | Case 3 |
|  | 200 | 100 | 50 | 200 | 100 | 50 |
|  | 50 | 25 | 15 | 1 | 0.2 | 0.1 |
| Accumulated measuring points $N_{\beta}$ $\sigma_{\beta}(\mathrm{rad})$ | Case 4 | Case 5 | Case 6 | Case 4 | Case 5 | Case 6 |
|  | 200 | 100 | 50 | 200 | 100 | 50 |
|  | $1 \times 10^{-3}$ | $1 \times 10^{-3}$ | $1 \times 10^{-3}$ | $1 \times 10^{-4}$ | $1 \times 10^{-4}$ | $1 \times 10^{-4}$ |

Table 10.5 Orbit elements of two satellites in frequency-only based passive orbit determination

| Description of orbit elements | Observing satellite (O) | Target satellite (T) |
| :--- | :---: | :---: |
| Semi-major axis $a(\mathrm{~km})$ | 42000 | 7100 |
| Eccentricity $e$ | 0.1 | 0.1 |
| Orbit inclination $i(\mathrm{deg})$ | 120 | 30 |
| RAAN $\Omega(\operatorname{deg})$ | 30 | 75 |
| Argument of perigee $\omega(\mathrm{deg})$ | 45 | 60 |
| Mean argument of perigee $M_{0}(\mathrm{deg})$ | 0 | 60 |

Table 10.6 Parameter search scopes and ISE setting

| Parameter/ orbit elements | Unknown $f_{0}$ |  |  | Known $f_{0}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Minimum | Maximum | ISE | Minimum | Maximum | ISE |
| Semi-major axis $a$ (km) | 6900 | 7500 | 100 | 7000 | 7200 | 10 |
| Eccentricity $e$ | 0 | 1 | 0.05 | 0.01 | 0.5 | 0.01 |
| Orbit inclination $i$ (deg) | 0 | 180 | 5 | 10 | 60 | 1 |
| RAAN $\Omega$ (deg) | 0 | 360 | 5 | 45 | 90 | 1 |
| Argument of perigee $\omega$ (deg) | 0 | 360 | 5 | 45 | 90 | 1 |
| Mean argument of perigee $M_{0}$ (deg) | 0 | 360 | 5 | 45 | 90 | 1 |

The performances of frequency-only based passive orbit determination in the case of the two-body model and considering $\mathrm{J}_{2}$ perturbation of earth oblateness through computer simulation are analyzed below.

### 10.5.1 Simulation in the Case of the Two-Body Model

The iterative search process of orbit elements of the target satellite at the epoch time by the method of the PSO algorithm is according to the simulation parameters in Table 10.4 and the


Figure 10.8 Particle swarms initialized by the PSO algorithm. (a) Results of the semi-major axis $a$ and eccentricity $e$, (b) results of the orbit inclination $i$ and RAAN $\Omega$, (c) results of the argument of perigee $\omega$ and the mean argument of perigee $M_{0}$, and (d) frequency measurement and fitting curve
parameter search scopes and ISE in Table 10.5. (Here only the iterative search conditions are given for unknown $f_{0}$ in the case of the two-body model as the iterative process in the case of known $f_{0}$ when considering orbit perturbation is similar.)
The corresponding results are shown in Figure 10.8 when particle swarms are initialized. In the figure, 'x' represents the location of every particle, 'o'" represents the current location of gBest, that is, the best estimation result up to the current iteration times, while '*'" represents the actual orbit elements of the satellite to be estimated. The vector of the state to be estimated is of six dimensions, so it is displayed in three figures, among which Figure 10.8a is the iteration results of the semi-major axis $a$ and eccentricity $e$, Figure 10.8 b is those of the orbit inclination $i$ and RAAN $\Omega$, Figure 10.8c is those of the argument of perigee $\omega$ and the epoch mean argument of perigee $M_{0}$, and Figure 10.8d is the frequency measurement result and the fitting curve of the PSO algorithm.
Results of particle swarms after 20 iterations are shown in Figure 10.9.
Results of particle swarms after 50 iterations are shown in Figure 10.10.
Results of particle swarms after 80 iterations are shown in Figure 10.11.
It can be seen from Figures 10.8 to 10.11 that with increasing iteration times, the difference between the orbit elements at the epoch time of the target satellite estimated by the method of


Figure 10.9 Particle swarms iterated for 20 times by the PSO algorithm. (a) Results of the semi-major axis $a$ and eccentricity $e$, (b) results of the orbit inclination $i$ and RAAN $\Omega$, (c) results of the argument of perigee $\omega$ and the mean argument of perigee $M_{0}$, and (d) frequency measurement and fitting curve
passive orbit determination based on PSO and the actual values reduces gradually while the fitting degree of corresponding frequency change curves becomes higher and higher.
Now the parameter estimation error of the frequency-only based passive orbit determination based on PSO algorithm will be calculated for unknown $f_{0}$ cases by statistical evaluation of Monte Carlo simulations. In order to evaluate the performance of passive orbit determination, the same satellite orbit elements as used in reference [1] will be taken for the BO passive tracking algorithm for state variables with the CRLB as reference. According to the simulation parameters shown in Table 10.4 and the parameter search scopes and ISEs shown in Table 10.4, the estimation errors of orbit elements of the target satellite under different conditions can be calculated and are shown in Figure 10.12 (the parameter estimation errors referred to in this chapter are all RMS errors of 50 Monte Carlo simulations).
The estimation error curve of the corresponding target signal carrier frequency $f_{0}$ and the curve of the fitness function values calculated according to expression (10.9) are shown in Figure 10.13. They all vary with the iteration times.


Figure 10.10 Particle swarms iterated 50 times by the PSO algorithm. (a) Results of the semi-major axis $a$ and eccentricity $e$, (b) results of the orbit inclination $i$ and RAAN $\Omega$, (c) results of the argument of perigee $\omega$ and the mean argument of perigee $M_{0}$, and (d) frequency measurement and fitting curve

The estimation errors of each parameter corresponding to several groups of simulation conditions above and the corresponding CRLB are shown in Figure 10.12 and Table 10.7.
According to Figure 10.12 and Table 10.7, the simulation results are as follows:

1. For unknown $f_{0}$ cases in the case of the two-body model, what has the largest effect on the estimation accuracy of frequency-only based passive orbit determination based on PSO is the accumulated points $N_{f}$ (length of the observation duration), followed by the frequency measurement error. The estimation error of the algorithm will increase with reduction of accumulated points.
2. The parameter estimation RMS error still does not reach the CRLB and it will deviate CRLB more and more with reduction of accumulated measuring points (length of the observation duration).
3. According to Figure 10.12, it can be seen that no matter how large the number of accumulated measuring points (here it refers to $N_{f}=200$ ) or how small the number of accumulated measuring points (here it refers to $N_{f}=50$ ) are, the parameter estimation accuracy of frequency-only based passive orbit determination based on PSO is better than that of the BO method in reference [1]. Here it must be specified that the BO method in reference [1] adopts the method of recursive filtering. For the frequency-only based passive orbit determination based on PSO in this chapter, the method of batch processing is adopted, so the performance comparison of the two methods are based on the same observation arc data. For example, when $N_{f}=100$, for the frequency-only based method based on PSO, the results of iter $_{\text {max }}$ times of observation data iterations at $N_{f}$ are compared with those derived according to the observation data at $N_{f}$ by the method of BO in reference [1]. As the comparison rules are therefore the same as those of the BO method in reference [1], they will be used in the following simulations in this chapter.


Figure 10.11 Particle swarms iterated 80 times by the PSO algorithm. (a) Results of the semi-major axis $a$ and eccentricity $e$, (b) results of the orbit inclination $i$ and RAAN $\Omega$, (c) results of the argument of perigee $\omega$ and the mean argument of perigee $M_{0}$, and (d) frequency measurement and fitting curve


Figure 10.12 Parameter estimation error curves for unknown $f_{0}$ cases in the case of the two-body model. Estimation error of (a) the semi-major axis $a$, (b) eccentricity $e$, (c) the orbit inclination $i$, (d) RAAN $\Omega$, (e) the argument of perigee $\omega$, and (f) the epoch mean argument of perigee $M_{0}$


Figure 10.13 Estimation error and fitness function variation curves for unknown $f_{0}$ cases in the case of the two-body model. (a) Fitness curve of $f_{0}$ and fitness curve of $X_{F O}$

Similarly, for known $f_{0}$ cases, the parameter estimation error of the frequency-only based passive orbit determination using the PSO algorithm can be calculated under the simulation scenarios and the parameter search scopes and ISEs through Monte Carlo simulations, as shown in Table 10.6. Here also the satellite orbit elements used in reference [1] are taken for state variables with the CRLB introduction derived in Section 10.4 as reference, with the corresponding estimation results. This is shown in Figure 10.14.
The fitness function values calculated according to expression (10.9) are shown in Figure 10.15 , which vary with the iteration times.
Estimation errors of each parameter corresponding to several groups of simulation conditions above and the corresponding CRLB are shown in Table 10.8.
Table 10.7 Estimation errors for unknown $f_{0}$ cases in the case of the two-body model

| Parameters/ <br> Orbit elements | Case 1 |  | Case 2 |  | Case 3 |  | Case 4 | Case 5 | Case 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Estimation error | CRLB | Estimation error | CRLB | Estimation error | CRLB | Estimation error | Estimation error | Estimation error |
| Semi-major axis $a$ (km) | 3.0724 | 0.0933 | 3.4991 | 0.5751 | 6.4472 | 2.5058 | 5.3374 | 8.3353 | 48.6343 |
| Eccentricity e | $4.6414 \times 10^{-4}$ | $6.2522 \times 10^{-5}$ | $7.0717 \times 10^{-4}$ | $1.0478 \times 10^{-4}$ | $1.7805 \times 10^{-3}$ | $1.3805 \times 10^{-4}$ | $1.1601 \times 10^{-3}$ | $2.0391 \times 10^{-3}$ | $8.7619 \times 10^{-3}$ |
| Orbit inclination $i\left({ }^{\circ}\right)$ | 0.0375 | 0.0014 | 0.0413 | 0.0040 | 0.0557 | 0.0121 | 0.0501 | 0.0697 | 0.1678 |
| RAAN $\Omega\left({ }^{\circ}\right.$ ) | 0.0673 | 0.0026 | 0.0764 | 0.0086 | 0.1012 | 0.0324 | 0.0910 | 0.1571 | 0.4063 |
| Argument of perigee $\omega\left({ }^{\circ}\right)$ | 0.0658 | 0.0323 | 0.0964 | 0.0414 | 0.2091 | 0.0615 | 0.1443 | 0.4736 | 0.8506 |
| Mean argument of perigee $M_{0}\left({ }^{\circ}\right)$ | 0.0572 | 0.0340 | 0.0953 | 0.0375 | 0.2258 | 0.0841 | 0.1484 | 0.5336 | 0.7642 |
| $f_{0}(\mathrm{~Hz})$ | 15.7 | 4.3 | 25.8 | 12.0 | 54.2 | 16.0 |  |  |  |



Figure 10.14 Parameter estimation error curves for known $f_{0}$ cases in the case of the two-body model. Estimation error of (a) the semi-major axis $a$, (b) eccentricity $e$, (c) the orbit inclination $i$, (d) RAAN $\Omega$, (e) the argument of perigee $\omega$, and (f) the epoch mean argument of perigee $M_{0}$

According to Figure 10.14 and Table 10.8, conclusions can be drawn from the simulation results:

1. For known $f_{0}$ cases and in the case of the two-body model, what has the largest effect on the parameter estimation accuracy of state is the accumulated measuring points $N_{f}$ (length of the observation arcs), followed by the frequency measurement error. Errors of the estimation algorithm will increase with reduction of the accumulated measuring points.
2. The estimation error in the case of known $f_{0}$ is closer to its CRLB than in the case of unknown $f_{0}$, which indicates that higher parameter estimation accuracy can be achieved after a certain amount of prior frequency information of the target is added.
3. According to Figure 10.14, it can be seen that when the number of accumulated points is small (here this refers to $N_{f}=50$ ), the estimation accuracy of passive orbit determination based on PSO is obviously better than that of the BO method in reference [1]; when the number of accumulated points grows (here this refers to $N_{f}=100$ ), the estimation accuracy of passive orbit determination based on PSO is slightly better than that of the BO method in reference [1]; and when the number of accumulated points grows to $N_{f}=200$, the estimation accuracy of the BO method in reference [1] is better than that of the passive orbit determination based on PSO.

According to Figures 10.12 and 10.15, it can be seen that in the case of the two-body model, the fitness function value for known $f_{0}$ cases is obviously smaller than that in the case of unknown $f_{0}$. The fitness function value is the sum of squares of the frequency estimation errors at each measuring point, which will become larger with an increasing number of accumulated measuring points, therefore causing the fitness function value of Case 1 in Figures 10.12 and 10.14 to be larger than those of Case 2 and Case 3. According to this effect, it can be concluded that the estimation accuracy for known $f_{0}$ cases is obviously better than that for unknown $f_{0}$ cases because the smaller the cost function value (which means that the better the fitness degree of frequency variation curves), the higher is the parameter estimation accuracy.
The computation load of frequency-only based passive orbit determination based on PSO and the grid search method $[16,17]$ is analyzed below. For simplicity, here only the different calculation steps of the two methods are compared. Assume that the computation load of every feasible solution to be calculated is 1 and the computation load by the method of PSO is $P \cdot$ iter $_{\text {max }}$, where $P$ is the particle number of the particle swarm and ite $_{\text {max }}$ is the maximum allowable iteration times. The computation load by the grid search method is $N_{1} \cdot N_{2} \ldots$. $N_{6}$, where $N_{l}$ is the grid number necessary for each dimension of parameters to be estimated in the grid search method ( $1 \leq l \leq 6$ ), and is determined by the expected resolution of each dimension of parameters in the grid search method.
Taking the initial search scopes of the simulation scenarios shown in Table 10.4 as an example, if we use the grid search method and the PSO algorithm for state estimations and their final estimation accuracies are required to be equivalent, the computation loads of the two methods are as shown in Table 10.9.
According to Table 10.9, it can be seen that it is rarely possible to use the computation load of the grid search method to obtain estimation accuracy equivalent to that of the PSO algorithm. Due to the different parameter estimation accuracies in the case of different accumulated measuring points (length of the observation arcs), whether it is for known $f_{0}$ cases or not, their computation loads will be different, with differences of the computation loads in various simulation scenarios within $10^{1}-10^{6}$. When the passive orbit determination algorithm based on
Table 10.8 Estimation errors for known $f_{0}$ cases in the case of the two-body model

| Parameters/ <br> Orbit elements | Case1 |  | Case 2 |  | Case 3 |  | Case 4 | Case 5 | Case 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Estimation error | CRLB | Estimation error | CRLB | Estimation error | CRLB | Estimation error | Estimation error | Estimation error |
| Semi-major axis $a$ (km) | 0.0914 | 0.0019 | 1.0525 | 0.0060 | 3.1059 | 0.1323 | 0.0910 | 1.8246 | 26.4322 |
| Eccentricity e | $6.0878 \times 10^{-5}$ | $1.2504 \times 10^{-6}$ | $3.6587 \times 10^{-4}$ | $8.4279 \times 10^{-6}$ | $1.7145 \times 10^{-3}$ | $2.3650 \times 10^{-5}$ | $4.6005 \times 10^{-6}$ | $3.0492 \times 10^{-3}$ | $6.7341 \times 10^{-3}$ |
| Orbit inclination $i\left({ }^{\circ}\right)$ | $1.7567 \times 10^{-3}$ | $2.7963 \times 10^{-5}$ | $6.0548 \times 10^{-3}$ | $1.1866 \times 10^{-4}$ | 0.0101 | $8.0397 \times 10^{-4}$ | $6.6002 \times 10^{-4}$ | 0.0132 | 0.0458 |
| RAAN $\Omega\left({ }^{\circ}\right.$ ) | $4.8744 \times 10^{-3}$ | $5.2008 \times 10^{-5}$ | $8.9316 \times 10^{-3}$ | $2.5215 \times 10^{-4}$ | 0.0291 | 0.0086 | $7.0091 \times 10^{-4}$ | 0.0176 | 0.0787 |
| Argument of perigee $\omega\left({ }^{\circ}\right)$ | $8.7549 \times 10^{-3}$ | $6.4537 \times 10^{-4}$ | 0.0228 | $2.7585 \times 10^{-3}$ | 0.0608 | 0.0174 | $3.5401 \times 10^{-3}$ | 0.0637 | 0.1805 |
| Mean argument of perigee $M_{0}\left({ }^{\circ}\right)$ | 0.0355 | $6.7978 \times 10^{-4}$ | 0.0604 | $3.5702 \times 10^{-3}$ | 0.0805 | 0.0192 | 0.0117 | 0.1212 | 0.2650 |



Figure 10.15 Fitness function variation curves for known $f_{0}$ cases in the case of the two-body model

Table 10.9 Analysis of computation load by the PSO algorithm and the grid search method

| Simulation scenario/ grid number | Unknown $f_{0}$ |  |  | Known $f_{0}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Case 1 | Case 2 | Case 3 | Case 1 | Case 2 | Case 3 |
| Semi-major axis $a$ (km) | 195 | 170 | 90 | 2180 | 190 | 64 |
| Eccentricity $e$ | 2150 | 1410 | 560 | 8040 | 1330 | 285 |
| Orbit inclination $i$ (deg) | 4800 | 4360 | 3230 | 28400 | 8250 | 4950 |
| RAAN $\Omega$ (deg) | 5340 | 4710 | 3550 | 9230 | 5030 | 1540 |
| Argument of perigee $\omega$ (deg) | 5470 | 3730 | 1720 | 5140 | 1970 | 740 |
| Mean argument of perigee $M_{0}(\mathrm{deg})$ | 6290 | 3770 | 1590 | 1260 | 740 | 560 |
| Total computation load of grid search | $3.7 \times 10^{20}$ | $6.9 \times 10^{19}$ | $1.6 \times 10^{18}$ | $2.9 \times 10^{22}$ | $1.5 \times 10^{19}$ | $5.7 \times 10^{16}$ |
| Computation load of the PSO algorithm | 16000 |  |  | 16000 |  |  |

PSO in this chapter is used, since it does not use the serial work method as used in the grid search method but uses the multidimensional parallel iteration searching method, it does not matter whether $f_{0}$ is known or not. Their computation loads only depend on the particle number of the particle swarm and the iteration times of the algorithm, and computation loads in the various simulation scenarios are the same and obviously smaller than those of the grid search method.


Figure 10.16 Parameter estimation error curves for unknown $f_{0}$ cases in the case of perturbation. Estimation error of (a) the semi-major axis $a$, (b) eccentricity $e$, (c) the orbit inclination $i$, (d) RAAN $\Omega$, (e) the argument of perigee $\omega$, and (f) the epoch mean argument of perigee $M_{0}$

Table 10.10 Estimation errors for unknown $f_{0}$ cases and in the case of $J_{2}$ perturbation

| Parameters/ orbit elements | Case 1 |  | Case 2 |  | Case 3 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Estimation error | CRLB | Estimation error | CRLB | Estimation error | CRLB |
| Semi-major axis $a$ (km) | 3.4333 | 0.0933 | 4.9681 | 0.5751 | 8.2837 | 2.5058 |
| Eccentricity $e$ | $9.7389 \times 10^{-4}$ | $6.2522 \times 10^{-5}$ | $1.8787 \times 10^{-3}$ | $1.0478 \times 10^{-4}$ | . $4955 \times 10^{-3}$ | $3805 \times 10^{-4}$ |
| Orbit inclination $i$ (deg) | 0.0368 | 0.0014 | 0.0449 | 0.0040 | 0.0660 | 0.0121 |
| RAAN $\Omega$ (deg) | 0.0675 | 0.0026 | 0.0812 | 0.0086 | 0.1277 | 0.0324 |
| Argument of perigee $\omega$ (deg) | 0.0726 | 0.0323 | 0.1264 | 0.0414 | 0.2688 | 0.0615 |
| Mean argument of perigee $M_{0}$ (deg) | 0.0722 | 0.0340 | 0.0964 | 0.0375 | 0.2956 | 0.0841 |
| $f_{0}(\mathrm{~Hz})$ | 17.2 | 4.3 | 33.2 | 12.0 | 78.4 | 16.0 |

### 10.5.2 Simulation in the Case of Considering the Perturbation

By Monte Carlo simulations, according to the simulation scenarios and parameter search scopes shown in the tables, the parameter estimation error of frequency-only based passive orbit determination in considering the perturbation is now calculated in this section. Since the known $f_{0}$ case is just a special case of the unknown $f_{0}$ case, only orbit determination under the case of the unknown $f_{0}$ is simulated here. For the convenience of comparison, here also the CRLB is taken as a reference (similar to the analysis in Chapter 8, the CRLB in the case of the two-body model is still adopted for the estimation error lower bound, as in the case of perturbation). With these parameters, the estimation results are shown in Figure 10.16. The estimation errors of each parameter in the case of several groups of simulation conditions above and the corresponding CRLB are shown in Table 10.10.
According to Figure 10.16 and Table 10.10, as well as the corresponding simulation results in the case of the two-body model, it can be concluded that:

1. When $J_{2}$ perturbation is considered, what has the most important effect on the parameter estimation accuracy of every state is still the accumulated measuring points $N_{f}$ (length of the observation arcs) followed by the frequency measurement error. The estimation error of the algorithm will increase with reduction of the accumulated measuring points.
2. Compared with the performance in the case of the two-body model, the parameter estimation accuracy when $J_{2}$ perturbation is considered drops slightly, but compared with the corresponding CRLB, the estimation error does not change obviously, indicating that even though when orbit perturbation is considered, a good state (orbit) estimation effect can still be obtained for the frequency-only based passive orbit determination based on PSO.

## References

1. Guo, F. and Fan, Y. (2005) A tracking method for satellite-to-satellite passive localization in space information confrontation. Journal of Astronautics, 26(2): P196-P200 (in Chinese).
2. Ho, K.C. and Xu, W.W. (2004) An accurate algebraic solution for moving source location using TDOA and FDOA measurements. IEEE Transactions on Signal Processing, 52 (9), 2453-2463.
3. Guo, F. (2008) A fixed single observer passive localization method based on multi-level grid search. Signal Processing, 24(6): P927-P930 (in Chinese).
4. Kay, S. (1998) Fundamentals of Statistical Signal Processing, Volume I: Estimation Theory/Volume II: Detection Theory, Prentice Hall PTR.
5. Poisel, R.A. (2012) Electronic Warfare Target Location Methods, 2nd edn, Artech House.
6. Gao, S. and Yang, J. (2006) Swarm Intelligence Algorithms and Applications. Beijing: China Water Power Press (in Chinese).
7. Cui, X. (2006) Multiobjective Evolutionary Algorithms and Their Applications. Beijing: National Defence Industry Press (in Chinese).
8. Zhang, L. (2005) Theory and Practice on the Particle Swarm Optimization Algorithm. Hangzhou: Zhejiang University (in Chinese).
9. Cristian, T.I. (2003) The particle swarm optimization algorithm: convergence analysis and parameter selection. Information Processing Letters, 85, 317-325.
10. Xi, X., Wang, W. and Gao, Y. (2003) Fundamentals of Near-Earth Spacecraft Orbit. Changsha: NUDT Publish House (in Chinese).
11. Xu, Y., Guo, F., and Feng, D. (2010) A single-satellite only TOA measured passive localization method. Journal of Astronautics, 2(31): P502-P507 (in Chinese).
12. Zhang, M., Feng, D. and Guo, F. (2009) Single-satellite passive localization based on DRC. Aerospace Electronic Warfare, 5(25) (in Chinese).
13. Ren, X. (1998) Orbit Dynamics of Sputnik. Changsha: NUDT Publish House (in Chinese).
14. Dennis, R. (1996) Satellite Communication, 3rd edn, McGraw-Hill Inc..
15. Liu, L. (2000) Orbit Theory of Spacecraft. Beijing: National Defence Industry Press (in Chinese).
16. Becker, K. (2005) Three-dimensional target motion analysis using angles and frequency measurements. IEEE Transactions on Aerospace and Electronic Systems, 41 (1), 284-301.
17. Becker, K. (1996) A general approach to TMA observability from angle and frequency measurements. IEEE Transactions on Aerospace and Electronic Systems, 32 (1), 487-494.

## 11

## A Prospect of Space Electronic Reconnaissance Technology

By summarizing and analyzing the history of the space electronic reconnaissance technology, its characteristics and trends are: composition of HEO (high earth orbit) satellites and LEO (low earth orbit) satellites, compensation of general monitoring and detailed survey, integration of multiple reconnaissance and monitoring tasks to realize the full time domain (7/24), and frequently reconnaissance and monitoring to the electronic targets all over the world so as to provide timely and effective information for military operation.

1. Multiple-satellite networking for military demands of both strategic and tactical reconnaissance

After a general survey and monitoring of important and hot spots have been ensured, tactical support can be enhanced by methods such as multiple-satellite networking and satellite-ground system design to effectively provide timely and effective information for major weapons and combat units.
2. Comprehensive reconnaissance for meeting the reconnaissance demands of the complex electromagnetic environment

The electronic reconnaissance satellite integrates multiple reconnaissance and monitoring tasks for radar, communication, measurement, control, navigation, and data links. It also has the capacity of photoelectric reconnaissance to meet the fast-changing demands of a high-tech battlefield in the future, enabling the battlefield commander to know the latest conditions on the battlefield in time, all day so as to make the correct decisions.
3. Composition of HEO satellites and LEO satellites for realizing the continuous monitoring, detailed survey, and precise measurement

HEO and LEO satellite reconnaissance is mainly used for long-time continuous general monitoring, detailed surveying, and accurately locating target emitters. Deployment of satellites in the future will include the reconnaissance network of multiple kinds of satellite orbits, such as LEO, MEO (medium earth orbit), HEO, and geosynchronous orbit, as well as providing continuous monitoring depending on HEO reconnaissance satellites and the
cooperative engagement capability to guide the LEO reconnaissance satellites to conduct further detailed surveys.
4. Diversified localization system for meeting the reconnaissance demands of different kinds of tasks

By direction finding technology of the interferometer, the single-satellite LOS (line-of-sight) localization with LEO satellites can meet the localization demands of many types of signals with high localization accuracy and strong adaptability to complex signal environment; by multiple-satellite TDOA (time difference of arrival) and FDOA (frequency difference of arrival) localization, ocean surveillance satellites can meet the relatively simple reconnaissance demands of ocean signal environments with high localization accuracy and long-time continuous monitoring; by HEO satellites using electronically scanning array LOS geolocation, continuous monitoring of signals with HEO and limited localization capability can be realized. Therefore, the localization system will be selected reasonably according to the tasks of the electronic reconnaissance satellites.
5. Synchronic development of large and small satellites for meeting the demands of long-term duty and emergency launching

Throughout the history of electronic reconnaissance satellites, it has been found that the electronic reconnaissance satellites are becoming heavier and their functions are becoming stronger, with their weights gradually growing from tens of kilograms earlier to 8 to -9 tons and their functions integrated with reconnaissance of various electronic signals such as radar, communication, remote measurement, and remote control. On the other hand, with the development of high technologies, such as microelectronics and lightweight materials, the developmental upsurge of small satellites of dozens to several kilograms has begun to spring up and the concept of their design has begun to emphasize single functions. Since these satellites have advantages, such as short preparation time, rapid renewal, low cost, and fast launching, they are especially applicable to the emergency launching in wartime, meeting the demands for short-term monitoring and reconnaissance of the battlefields and conflict areas.

## Appendix

## Transformation of Orbit Elements, State and Coordinates of Satellites in Two-Body Motion

In localization, there are often transformations of orbit elements ( $a, e, i, \Omega, \omega, \tau$ ), motion states of satellites $(r, \alpha, \delta, v, \Theta, A)$, and the satellite position vector $[X, Y, Z]$ and velocity vector $[\dot{X}, \dot{Y}, \dot{Z}]$ on the coordinate of the ECI (earth centered inertial) coordinates, among which, one-to-one mapping relations are formed in two-body motion. For the convenience of readers, here the transformation relations are set out specially.

1. Obtain the state of orbits according to the orbit elements

To obtain the state $(r, \alpha, \delta, v, \Theta, A)$ at time $t$ according to the orbit elements ( $a, e, i$, $\Omega, \omega, \tau$ ), the following processes will be used.Firstly, calculate the eccentric anomaly $E$ according to

$$
\begin{equation*}
E-e \sin E=\left(\frac{\mu}{a^{3}}\right)^{1 / 2}(t-\tau) . \tag{A.1}
\end{equation*}
$$

Equation (A.1) is a transcendental equation, so it can only be solved by numerical solution, that is, $E$ can be obtained through iterative calculation. The expression $\mu=398603 \mathrm{~km}^{3} / \mathrm{s}^{2}$ is the gravitational constant (Kepler constant).

After the eccentric anomaly $E$ is obtained according to expression (A.1), the true anomaly $f$ can be obtained according to the following expression:

$$
\begin{equation*}
f=2 \tan ^{-1}\left[\left(\frac{1+e}{1-e}\right)^{1 / 2} \operatorname{tg} \frac{E}{2}\right] \tag{A.2}
\end{equation*}
$$

After the true anomaly $f$ is obtained, the following can be calculated according to the expressions below:

$$
\begin{equation*}
u=\omega+f \tag{A.3}
\end{equation*}
$$

$$
\left.\begin{array}{rl}
\delta & =\arcsin (\sin i \sin u) . \\
\cos (\alpha-\Omega) & =\frac{\cos u}{\cos \delta} \\
\sin (\alpha-\Omega) & =\tan \delta \cot i
\end{array}\right\},
$$

2. Obtain the orbit elements according to the state of orbits.

If the state at a certain time $(r, \alpha, \delta, v, \Theta, A)$ is known, the orbit elements $(a, e, i, \Omega, \omega$, $\tau)$ can be obtained according to the expressions below:

$$
\left.\begin{array}{rl}
a & =\frac{\mu r}{2 \mu-r v^{2}} . \\
e \sin E & =\frac{r v \sin \Theta}{\sqrt{\mu a}} \\
e \cos E & =1-\frac{r}{a}
\end{array}\right\}, ~ \begin{aligned}
e & =\left(e^{2} \cos ^{2} E+e^{2} \sin ^{2} E\right)^{1 / 2} \\
E & =\operatorname{arctg} \frac{e \sin E}{e \cos E} \\
\tau & =t-\left(\frac{a^{3}}{\mu}\right)^{1 / 2}(E-e \sin E), \\
i & =\arccos (\cos \delta \sin A), \\
\sin (\alpha-\Omega) & =\operatorname{tg} \delta \operatorname{ctg} i \\
\cos (\alpha-\Omega) & \left.=\frac{\cos A}{\sin i}\right\},  \tag{A.19}\\
\Omega & =\alpha-\operatorname{arctg}\left(\frac{\sin (\alpha-\Omega)}{\cos (\alpha-\Omega)}\right), \\
\{\sin u & =\frac{\sin \delta}{\sin i} \\
\cos u & =\operatorname{ctgictg} A
\end{aligned}
$$

$$
\begin{align*}
& u=\operatorname{arctg} \frac{\sin u}{\cos u}  \tag{A.20}\\
& f=2 \operatorname{arctg}\left[\left(\frac{1+e}{1-e}\right)^{1 / 2} \operatorname{tg} \frac{E}{2}\right]  \tag{A.21}\\
& \omega=u-f \tag{A.22}
\end{align*}
$$

3. Obtain the state of the orbit according to the ECI coordinate of the satellite.

According to the geometric relations, it can be found that:

$$
\left.\begin{array}{rl}
r & =\sqrt{X^{2}+Y^{2}+Z^{2}} \\
\alpha & =\tan ^{-1}\left(\frac{Y}{X}\right) \\
\delta & =\tan ^{-1}\left(\frac{Z}{\sqrt{X^{2}+Y^{2}}}\right)  \tag{A.23}\\
v & =\sqrt{\dot{X}^{2}+\dot{Y}^{2}+\dot{Z}^{2}} \\
\Theta & =\arcsin \left(\frac{X \dot{X}+Y \dot{Y}+Z \dot{Z}}{r v}\right) \\
A & =\tan ^{-1}\left(\frac{-\dot{X} \sin \alpha+\dot{Y} \cos \alpha}{-\sin \delta \cos \alpha \dot{X}-\sin \delta \sin \alpha \dot{Y}+\cos \delta \dot{Z}}\right)
\end{array}\right\}
$$

4. Obtain the satellite coordinate in the ECI system according to the state of the orbit.

The expressions to obtain the position vector $[X, Y, Z]$ and the velocity vector $[\dot{X}, \dot{Y}, \dot{Z}]$ of the satellite according to the motion state are

$$
\left.\begin{array}{rl}
X & =r \cos \alpha \cos \delta \\
Y & =r \sin \alpha \cos \delta \\
Z & =r \sin \delta \\
\dot{X} & =v(\cos \alpha \cos \delta \sin \Theta-\sin \alpha \cos \Theta \sin A-\sin \delta \cos \alpha \cos \Theta \cos A)  \tag{A.24}\\
\dot{Y} & =v(\sin \alpha \cos \delta \sin \Theta+\cos \alpha \cos \Theta \sin A-\sin \delta \sin \alpha \cos \Theta \cos A) \\
\dot{Z} & =v(\sin \delta \sin \Theta+\cos \delta \cos \Theta \cos A)
\end{array}\right\} .
$$

## Index

Added Guassian white noise, 40
Altitude, 16, 22, 27
Altitude assumption, 100, 112, 215
Altitude error, 60, 112, 215
Altitude input location algorithm, 100
Ambiguity problem, 51, 102
Ambiguous geolocation point, 7, 134, 143
Amplitude-comparison, 48
Analytical solution, 117, 315
Angle of arrival(AOA), 47, 219
Antenna, 10, 30, 48
Apogee, 14
Approximate analytical method, 141
Array, 55
Auxiliary stations, 211

Band, 2
Baseline, 6, 50, 123, 193
Bearings only(BO), 45, 64, 227
Best linear unbiased estimation, 67
Bias, 38, 113
Binary phase-shift keying(BPSK), 166

Calibration, 117, 159
Central limit theorem, 40
Circular error probability, 41
Circular orbit, 15
Combined method, 261
Communication intelligence, 2
Companion matrix, 140

Computation load, 166, 170, 345
Condition number, 235
Constellation, 103
Conventional inertial system, 21
Conventional terrestrial pole, 28
Conventional terrestrial system, 28
Coverage, 30
Cramér-Rao lower bound(CRLB), 60, 85, 165, 180
Cross ambiguity function(CAF), 165

Digital signal processor(DSP), 56, 171
Direction finding(DF), 47
Direction of arrival(DOA), 47, 144
Discrete Fourier transform(DFT), 55
Doppler, 6, 134
Doppler rate of changing, 179
Dual-satellite TDOA-FDOA, 5, 133

Earth centered inertial, 21
Earth geodetic coordinates system, 28
Earth oblateness, 232
Earth-center earth-fixed coordinates, 22
Eccentricity, 13, 15
Eigenvalues, 41
Electronic intelligence, 1, 10
Electronic reconnaissance System, 3
Electronic warfare(EW), 49
Ellipsoid model, 59
Elliptical error probable, 42

Elliptical orbit, 14
Emitter, 2
Epoch mean argument of perigee, 21
Error, 38
Error ellipse probability, 42
Evolutionary algorithm, 317
Extended Kalmanfilter(EKF), 65, 221
Extremely high frequency, 2

Fast Fourier transform, 171
Field-programmable gate array, 171
Fisher information matrix, 60, 85, 180, 330
Five-station calibration geolocation algorithm, 163
Four-station calibration method, 161
Frequency changing rate, 183
Frequency difference of arrival(FDOA), 5, 133
Frequency modulation, 187
Frequency of arrival, 183
Frquency, 2, 135

Gauss-Newton, 138
Geodetic Cartersian coordinates, 22
Geographical information system, 100
Geolocation, 2
Geometric dilution of precision, 40, 110
Geostationary orbit, 15
Geosynchronous orbit, 16
Global positioning system(GPS), 23, 112

High earth orbit, 2
Higher order terms, 157, 230
Highly elliptical orbit, 15
Hyperboloid, 6, 36

Inclination, 15
Independent and identically distributed, 39
Initial state error, 239
Instantaneous velocity, 17, 134
Intercept, 1
Interferometer, 49
Intermediate frequency, 135
Intersection, 6, 37, 141
Iterative method, 29, 136, 307

J2 perturbation, 277
J2000.0 geocentric coordinates system, 21
Jacobian matrix, 44, 61, 231

Kepler's laws, 13
Kinematic principle, 177
Least-squares(LS), 67, 84, 179
Line of position, 34
Line of sight, 3, 47
Local oscillator, 155
Long baseline interferometer, 52, 200
Low earth orbit, 16
Low noise amplifier, 10
Lowpass filter, 50

Maximum likelihood estimation, 84
Mean anomaly, 19
Measurement model, 218
Medium earth orbit, 16
Modefied gain extended Kalman filter, 221
Multi-baseline interferometer, 53
Multiple signal classification, 56
Multi-satellite TDOA geolocation, 7, 79

Navigation, 112
Near space, 203
Newton iteration method, 235
Nonlinear least-squares, 68, 195
Non-uniform linear array, 56
North-East-Down(NED), 23
No-solution problems, 102

Observability, 44
Optimal geometric configuration, 207
Orbit, 13
Orbit determination, 227
Orbit elements, 18, 351
Orbit ellipse, 14
Orbital period, 14
Osculation error, 86

Particle, 195, 317
Particle filter, 195
Particle swarm optimization, 317

Passive orbit determination, 227
Perigee, 13
Phase ambiguity, 51
Phase difference, 50
Phase rate of changing, 192
Phase shift keying, 166
Pitch, 24
Platform, 203
Platform body coordinates system, 24
Polar orbit, 15
Primary station, 216
Prior knowledge, 79
Pseudoinverse, 67
Pulse repetition frequency(PRF), 168
Pulse repetition interval, 168
Radio, 228
Radio frequency(RF), 2
Reciver, 10
Recursive orbit, 16
Reference sources, 155
Regular spherical model, 27, 87
Resolution, 103
Right ascension of the ascening Node, 19
Roll, 24
Root mean square(RMS), 38
Root mean square error(RMSE), 39

Satellite, 13
Satellite body coordinates system, 24
Satellite tool kit(STK), 67
Semi-major axis, 14
Side reconnaissance coverage area, 33
Signal Intelligence(SIGINT), 2
Signal-to-noise ratio(SNR), 33, 74, 165
Space electronic reconnaissance, 1

Spherical error probable(SEP), 41
Spherical iteration method, 92
Standard error, 38
State model, 228, 262
Storage, 10
Sub-satellite point, 30
Sunsynchronous orbit, 16
System observability, 44

Taylor series, 157, 229
Telemetry, tracking, command and monitoring(TTC\&M), 10
Three-satellite geolocation, 7, 79
Three-station calibration method, 117
Time difference of arrival(TDOA), 5, 211
Time of arrival, 80, 212
Time past perigee, 20, 313
Topocentric-horizon coordinates system, 23
Tracking and data relay satellite sytem, 10
Triangulation, 65, 204
Two-body model, 227

Unambiguous visual angle, 51
Uniform linear array, 53
Universal gravitational constant, 100
Unmanned aerial vehicle(UAV), 204
Very high frequency(VHF), 2
Video frequency(VF), 2
Weighted least-squares, 67
WGS-84 earth ellipsoid model, 28
World geodetic system, 27
Yaw, 24

# Space Electronic Reconnaissance 

# Localization Theories and Methods 

Fucheng Guo, National University of Defense Technology, P.R. China Yun Fan, National University of Defense Technology, P.R. China Yiyu Zhou, National University of Defense Technology, P.R. China Caigen Zhou, National University of Defense Technology, P.R. China Qiang Li, National University of Defense Technology, P.R. China

Determining the positions of various radar, communication, or navigation sources by intercepting radio signals transmitted from these sources is very useful in electronic intelligence collection and early warning. Due to the regular orbit of a satellite and the prior knowledge of emitters on the Earth's surface, the localization problem in space electronic reconnaissance is intrinsically different from geolocation problems using platforms on land, ocean or air. This book presents some basic theories and methods of how to geolocate the emitter on earth or in aerospace by using one or multiple satellites.

- Presents the theories and methods of determining a source's position in space through the use of satellites.
- Introduces the methods, principles and technologies of
- localization and tracking sources with space electronic reconnaissance systems
- localization algorithms and error in satellite system and near-space platform systems
- tracking algorithms and error in single satellite-to-satellite tracking systems.
- Provides the fundamentals, mathematics, analysis, measurements, and systems of localization with emphasis on defense industry applications.

This book is written for engineers and researchers working in avionics, radar, communication, navigation and electronic warfare. The book can also be used by postgraduates studying aerospace engineering, electronic engineering, communication engineering, and electronic countermeasures.


[^0]:    Space Electronic Reconnaissance: Localization Theories and Methods, First Edition.
    Fucheng Guo, Yun Fan, Yiyu Zhou, Caigen Zhou and Qiang Li.
    © 2014 John Wiley \& Sons Singapore Pte Ltd. Published 2014 by John Wiley \& Sons Singapore Pte Ltd.

[^1]:    Space Electronic Reconnaissance: Localization Theories and Methods, First Edition.
    Fucheng Guo, Yun Fan, Yiyu Zhou, Caigen Zhou and Qiang Li.
    © 2014 John Wiley \& Sons Singapore Pte Ltd. Published 2014 by John Wiley \& Sons Singapore Pte Ltd.

[^2]:    Space Electronic Reconnaissance: Localization Theories and Methods, First Edition.
    Fucheng Guo, Yun Fan, Yiyu Zhou, Caigen Zhou and Qiang Li.
    © 2014 John Wiley \& Sons Singapore Pte Ltd. Published 2014 by John Wiley \& Sons Singapore Pte Ltd.

[^3]:    Space Electronic Reconnaissance: Localization Theories and Methods, First Edition.
    Fucheng Guo, Yun Fan, Yiyu Zhou, Caigen Zhou and Qiang Li.
    © 2014 John Wiley \& Sons Singapore Pte Ltd. Published 2014 by John Wiley \& Sons Singapore Pte Ltd.

